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Twisted boundary states and representation of generalized fusion algebra

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The mutual consistency of boundary conditions twisted by an automorphism group G of the chiral algebra is studied for general modular invariants of rational conformal field theories. We show that a consistent set of twisted boundary states associated with any modular invariant realizes a non-negative integer matrix representation (NIM-rep) of the generalized fusion algebra, an extension of the fusion algebra by representations of the twisted chiral algebra associated with the automorphism group G. We check this result for several concrete cases. In particular, we find that two NIM-reps of the fusion algebra for $su(3)_k\,(k=3,5)$ are organized into a NIM-rep of the generalized fusion algebra for the charge-conjugation automorphism of $su(3)_k$. We point out that the generalized fusion algebra is non-commutative if G is non-abelian and provide some examples for $G \cong S_3$. Finally, we give an argument that the graph fusion algebra associated with simple current extensions coincides with the generalized fusion algebra for the extended chiral algebra, and thereby explain that the graph fusion algebra contains the fusion algebra of the extended theory as a subalgebra.

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1 Introduction

The classification of conformally invariant boundary states in two-dimensional conformal field theories (CFTs) is an interesting subject of research, both from the point of view of purely field theoretical problems and applications to condensed matter physics or string theory. In string theory, conformally invariant boundary states correspond to D-branes, which are considered to be submanifolds embedded in the target space of strings. The classification of boundary states is therefore the classification of D-branes, which might be helpful in understanding the nature of stringy geometry.

This classification problem of boundary states is hard to treat in full generality, and one would need some restriction on the problem to make it tractable. A natural choice for such a restriction is to consider only rational CFTs (RCFTs) [1] and boundary states that preserve the full chiral algebra of RCFTs. Actually, in a RCFT, every consistent set of boundary states should satisfy a set of simple algebraic equations, the so-called Cardy equation [1], if the full chiral algebra is preserved in the open-string channel. Finding a set of Cardy states, *i.e.*, the states satisfying the Cardy equation, is equivalent to finding a non-negative integer matrix representation (NIM-rep) of the fusion algebra [2][2] assuming that the set of boundary states is 'complete' in a sense of [3]. The classification of boundary states for this case is therefore related to the classification of NIM-reps of the fusion algebra [7].

Clearly, the boundary states preserving the full chiral algebra constitute only a subset of conformally invariant boundary states, since the chiral algebra in general contains the Virasoro algebra as a proper subalgebra. In order to obtain, and classify, more states other than those preserving the full chiral algebra, we need some way to reduce the symmetry preserved by boundary states in a controlled manner. One way to accomplish this is to modify (or 'twist') the boundary condition of the chiral currents by an automorphism group G of the chiral algebra. The corresponding 'twisted boundary states' preserve the Virasoro algebra if G keeps it fixed. The states preserving the full chiral algebra (the 'untwisted' states) correspond to the identity element of G and are considered to be a particular case of the twisted states. In this sense, the twisted states associated with the automorphism group G realize a generalization of untwisted states.

The classification problem of twisted boundary states is systematically studied by several groups [8, 9, 10, 11]. [3] In particular, it is pointed out in [10] (see also [9, 15]) that the mutual consistency of twisted boundary states in the charge-conjugation modular invariant follows from the integrality of the structure constants of a certain algebra called *the generalized fusion algebra*. This algebra is defined by the fusion of representations of the twisted chiral algebra and hence contains, as a subalgebra, the ordinary fusion algebra consisting of only representations of the (untwisted) chiral algebra. This relation of twisted boundary states with the generalized fusion algebra is quite analogous to the relation of untwisted boundary states with the ordinary fusion algebra, and suggests that the generalized fusion algebra also

[1] For the current status of the research about non-rational cases, see *e.g.* [21].

[2] It should be noted that the Cardy condition is a necessary condition on boundary states [4, 5, 6]. Actually, there is a NIM-rep which does not give rise to any consistent boundary states, such as the tadpole NIM-rep of $su(2)_{2n-1}$ (see *e.g.* [2, 7]).

[3] For the other approaches to boundary states with less symmetry, see, *e.g.*, [12, 13, 14].

plays some role in the classification problem of twisted boundary states.

In this paper, we give an answer to the above question concerning the role of the generalized fusion algebra in the classification problem of twisted boundary states. Namely, we show that a set of mutually consistent twisted boundary states realizes *a NIM-rep of the generalized fusion algebra*. Our result is valid for any finite automorphism group including non-abelian ones. We point out that the generalized fusion algebra for a non-abelian automorphism group is *non-commutative* and provide some examples for the case of the symmetric group S_3. We develop the representation theory of the generalized fusion algebras and find their irreducible representations, which generalize quantum dimensions for the case of the ordinary fusion algebras. Unlike quantum dimensions, irreducible representations of the generalized fusion algebras are possible to have dimension greater than one, reflecting the non-commutativity of the generalized fusion algebras. We also point out that a NIM-rep of the generalized fusion algebra is decomposed into NIM-reps of the ordinary fusion algebra, since the former contains the latter as a subalgebra. As a check of our argument, we explicitly construct twisted boundary states in several concrete cases and show that their overlap matrices indeed form a NIM-rep of the corresponding generalized fusion algebra. In particular, we find that two NIM-reps of $su(3)_k$ ($k = 3, 5$) are combined into a NIM-rep of the generalized fusion algebra of $su(3)_k$ for the automorphism group $\{1, \omega_c\}$, where ω_c is the charge-conjugation automorphism of $su(3)_k$. We also give an argument that some graph fusion algebra [19, 20, 2] can be regarded as the generalized fusion algebra. More precisely, we show that the graph fusion algebra associated with boundary states in a simple current extension [16] coincides with the generalized fusion algebra of the extended chiral algebra for an appropriate automorphism group. This result naturally explains the observation [19] that the graph fusion algebra contains the fusion algebra of the extended theory as a subalgebra.

An algebraic structure of twisted boundary states is also studied in [8, 9] and we should clarify the difference of our analysis from that of [8, 9]. In [8], the case that a chiral algebra $\mathcal{A}$ is obtained by a simple current extension is considered. In this setting, the unextended chiral algebra $\mathcal{A}_0$ is characterized as the fixed point algebra of some finite abelian automorphism group G of $\mathcal{A}$, which is the dual of the simple current group used in the extension. The authors of [8] classify boundary states that preserve $\mathcal{A}_0$ in the charge-conjugation invariant of $\mathcal{A}$ using simple current techniques, and show that the states preserving $\mathcal{A}_0$ can be regarded as the G-twisted states. They also show that the G-twisted boundary states correspond to one-dimensional representations of some commutative algebra called the classifying algebra [17]. In [9], the structure of twisted boundary states in the charge-conjugation invariant is investigated for an arbitrary chiral algebra, in particular untwisted affine Lie algebras, with the automorphism group $G \cong \mathbb{Z}_2$, and the corresponding generalized fusion algebra together with the classifying algebra are studied. Our stance on the problem is slightly different from these studies. We do not intend to construct twisted boundary states explicitly; rather we clarify a consistency condition of twisted states assuming their existence. We impose no restrictions on the model we consider. The charge-conjugation modular invariant, as well as an abelian automorphism group, is a particular case of our analysis. Finally, the expression [9] for the case of the $\mathbb{Z}_2$ automorphism suggests that the classifying algebra in [8] is the dual of the generalized fusion algebra for abelian automorphism groups in the sense of C-algebras

[18, 19, 20], although we have no proof for this statement. Since this duality exchanges the irreducible representations with the elements of the algebra, the result of [8] is consistent with the observation [10, 15] that the twisted boundary states in the charge-conjugation invariant realize the generalized fusion algebra, which is the starting point of our analysis.

This paper is organized as follows. In the next section, we give the definition of the generalized fusion algebras using the twisted boundary states in the charge-conjugation modular invariant along the lines of [10]. The case of the affine Lie algebra $so(8)_1$ and its triality automorphism group is treated in detail. In Section 3, we show that the fusion coefficients of the generalized fusion algebras can be expressed in a form similar to the Verlinde formula even in the case of non-abelian automorphisms, and obtain the irreducible representations of the generalized fusion algebras. Based on this result, we show in Section 4 that the overlap matrices of mutually consistent twisted boundary states in any modular invariant form a NIM-rep of the generalized fusion algebra. Some examples including the case of non-abelian automorphisms are presented in Section 5. In Section 6, we argue the relation of graph fusion algebras with the generalized fusion algebras. The final section is devoted to discussions. In the appendices, we present the details of the calculation in our examples. In Appendix A, we give an explicit realization of the algebra embedding $su(3)_3 \subset so(8)_1$ and show that the charge-conjugation automorphism group $\{1, \omega_c\}$ of $su(3)$ has a lift to $so(8)$, which is isomorphic to the symmetric group S_3. In Appendix B, the generalized fusion algebra of $su(3)_k \, (k = 3, 5)$ for the charge-conjugation automorphism group is determined and it is shown that the twisted boundary states associated with non-trivial modular invariants yield a NIM-rep of the generalized fusion algebra. The same calculation is done in Appendix C for $su(3)_1^{\oplus 3}$ and its permutation automorphism of three factors.

2 Generalized fusion algebra

In this section, we give a definition of the generalized fusion algebra for the chiral algebra $\mathcal{A}$ with the automorphism group G by using the cylinder amplitude for the (twisted) boundary states in the charge-conjugation modular invariant of $\mathcal{A}$.

2.1 Untwisted boundary states and ordinary fusion algebra

We start from the construction of the untwisted boundary states and their relation with the ordinary fusion algebra [1].

Let $\mathcal{I}$ be the set of all the irreducible representations of the chiral algebra $\mathcal{A}$. Since we consider only rational cases, $\mathcal{I}$ is a finite set. The vacuum representation of $\mathcal{A}$ is denoted by $0 \in \mathcal{I}$. For each $\lambda \in \mathcal{I}$, the character $\chi_\lambda(q)$ of the representation λ is defined as

$$\chi_\lambda(q) = \mathrm{Tr}_{\mathcal{H}_\lambda} \, q^{L_0 - \frac{c}{24}} \,, \tag{2.1}$$

where $q = e^{2\pi i \tau}$, c is the central charge of the theory and the trace is taken in the irreducible representation space $\mathcal{H}_\lambda$ with the highest-weight λ. Under the modular transformation

$\tau \mapsto -1/\tau$, the characters $\chi_\lambda(q)$ transform as follows

$$\chi_\lambda(q) = \sum_{\mu \in \mathcal{I}} S_{\lambda\mu} \chi_\mu(\tilde{q}) \quad (\tilde{q} = e^{-2\pi i/\tau})\,. \tag{2.2}$$

The modular transformation matrix S is unitary and symmetric. The (ordinary) fusion algebra is an associative commutative algebra defined by the fusion of two representations in $\mathcal{I}$,

$$(\lambda) \times (\mu) = \sum_{\nu \in \mathcal{I}} \mathcal{N}_{\lambda\mu}{}^{\nu} (\nu) \qquad (\lambda,\, \mu \in \mathcal{I})\,, \tag{2.3}$$

where the fusion coefficients $\mathcal{N}_{\lambda\mu}{}^{\nu}$ take values in non-negative integers and are related with the modular transformation matrix via the Verlinde formula [22],

$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \sum_{\sigma \in \mathcal{I}} \frac{S_{\lambda\sigma} S_{\mu\sigma} \overline{S_{\nu\sigma}}}{S_{0\sigma}}\,. \tag{2.4}$$

A boundary state $|\alpha\rangle$ is a coherent state in the closed-string channel that preserves a half of the (super) conformal symmetry on the worldsheet. We also require that $|\alpha\rangle$ keep the full chiral algebra $\mathcal{A}$. In terms of modes, the boundary condition can be written as

$$(W_n - (-1)^h \widetilde{W}_{-n})|\alpha\rangle = 0\,, \tag{2.5}$$

where W_n and $\widetilde{W}_{-n}$ are respectively the modes of the current $W(z)$ of the holomorphic chiral algebra $\mathcal{A}$ with conformal dimension h and its counterpart $\widetilde{W}(\bar{z})$ of the anti-holomorphic chiral algebra $\widetilde{\mathcal{A}}$. For each $\lambda \in \mathcal{I}$, we can construct a building block $|(\lambda, \lambda^*)\rangle\rangle$ of boundary states satisfying (2.5),

$$|(\lambda, \lambda^*)\rangle\rangle = \frac{1}{\sqrt{S_{0\lambda}}} \sum_N |\lambda; N\rangle \otimes \overline{|\lambda; N\rangle} \, \in \mathcal{H}_\lambda \otimes \widetilde{\mathcal{H}}_{\lambda^*}\,, \tag{2.6}$$

which is called the Ishibashi state [23]. Here $|\lambda; N\rangle$ is an orthonormal basis of the representation space $\mathcal{H}_\lambda$ and we denote by $\lambda^* \in \mathcal{I}$ the conjugate representation of $\lambda \in \mathcal{I}$. This notation for Ishibashi states is slightly different from that used in other references, such as $|\lambda\rangle\rangle$. We adopt $|(\lambda, \lambda^*)\rangle\rangle$, instead of $|\lambda\rangle\rangle$, to indicate explicitly that this state is composed of the states in $\mathcal{H}_\lambda \otimes \widetilde{\mathcal{H}}_{\lambda^*}$ with the highest-weight state $|\lambda\rangle \otimes |\lambda^*\rangle$ of $\mathcal{A} \times \widetilde{\mathcal{A}}$. Since the boundary condition (2.5) is linear, multiplying $|(\lambda, \lambda^*)\rangle\rangle$ by some constant still gives a solution to (2.5). The above normalization (2.6) is chosen so that the following equation is satisfied,

$$\langle\langle(\lambda, \lambda^*)|\tilde{q}^{\frac{1}{2}H_c}|(\lambda', \lambda'^*)\rangle\rangle = \delta_{\lambda\lambda'} \frac{1}{S_{0\lambda}} \sum_N \langle\lambda; N|\tilde{q}^{L_0 - \frac{c}{24}}|\lambda; N\rangle = \delta_{\lambda\lambda'} \frac{1}{S_{0\lambda}} \chi_\lambda(\tilde{q})\,. \tag{2.7}$$

Here H_c is the closed string Hamiltonian

$$H_c = L_0 + \tilde{L}_0 - \frac{c}{12} \tag{2.8}$$

and we used $(L_0 - \tilde{L}_0)|(\lambda, \lambda^*)\rangle\rangle = 0$ to eliminate $\tilde{L}_0$.

In order to find solutions to (2.5), we have to specify the spectrum of bulk fields, or equivalently the modular invariant

$$Z = \sum_{\lambda,\mu\in\mathcal{I}} M_{\lambda\mu}\chi_\lambda\overline{\chi_\mu}\,, \tag{2.9}$$

where $\chi_\lambda\overline{\chi_\mu}$ corresponds to a bulk field $\Phi_{(\lambda,\mu^*)}$ which carries a representation (λ,μ^*) of $\mathcal{A}\times\widetilde{\mathcal{A}}$. In (2.9), there are $M_{\lambda\mu}$ fields with representation (λ,μ^*) and we should distinguish them by putting an index like $\Phi_{(\lambda,\mu^*)_i}$ if $M_{\lambda\mu} > 1$. In order to keep the notation simple, however, we omit the extra index i and simply use $\Phi_{(\lambda,\mu^*)}$ understanding that they appear in (2.9) with the multiplicity $M_{\lambda\mu}$. We denote by $\mathrm{Spec}(Z)$ the set of labels for bulk fields available in (2.9),

$$\mathrm{Spec}(Z) = \{(\lambda,\mu^*)\,|\,\Phi_{(\lambda,\mu^*)} \text{ exists in (2.9)}\}\,. \tag{2.10}$$

Note that a symbol (λ,μ^*) appears $M_{\lambda\mu}$ times in $\mathrm{Spec}(Z)$ corresponding to $M_{\lambda\mu}$ independent bulk fields with the label (λ,μ^*). In this section, we restrict ourselves to the case of the charge-conjugation modular invariant

$$Z_c = \sum_{\lambda\in\mathcal{I}}|\chi_\lambda|^2\,, \tag{2.11}$$

for which the spectrum of bulk fields reads

$$\mathrm{Spec}(Z_c) = \{(\lambda,\lambda^*)\,|\,\lambda\in\mathcal{I}\}\,. \tag{2.12}$$

We have one Ishibashi state $|(\lambda,\lambda^*)\rangle\rangle$ for each bulk field $\Phi_{(\lambda,\lambda^*)}$. A general boundary state $|\alpha\rangle$ satisfying (2.5) are therefore a linear combination of $|(\lambda,\lambda^*)\rangle\rangle$ $(\lambda\in\mathcal{I})$,

$$|\alpha\rangle = \sum_{\lambda\in\mathcal{I}}\Psi_{\alpha(\lambda,\lambda^*)}|(\lambda,\lambda^*)\rangle\rangle\,. \tag{2.13}$$

A consistent set of boundary states is obtained by considering the cylinder amplitude $Z_{\alpha\beta}$ with boundary conditions α and β. This amplitude can be calculated in two ways: a closed-string propagation between two boundary states, and an open-string one-loop amplitude. In the closed-string channel, the cylinder amplitude can be expressed as

$$Z_{\alpha\beta} = \langle\beta|\tilde{q}^{\frac{1}{2}H_c}|\alpha\rangle\,, \tag{2.14}$$

where $\tilde{q} = e^{-2\pi/t}$ and t is the circumference of the cylinder. On the other hand, the calculation in the open-string channel yields

$$Z_{\alpha\beta} = \sum_{\lambda\in\mathcal{I}}(n_\lambda)_\alpha{}^\beta\,\chi_\lambda(q)\,, \tag{2.15}$$

where $q = e^{-2\pi t}$ and $(n_\lambda)_\alpha{}^\beta$ is the multiplicity of representation λ in the spectrum of open string with boundary condition $[\beta,\alpha]$. Note that this form of open-string spectrum,

$\bigoplus_{\lambda\in\mathcal{I}}(n_\lambda)_\alpha{}^\beta\,\mathcal{H}_\lambda$, follows from the requirement that the boundary conditions α and β of open string preserve the chiral algebra $\mathcal{A}$. Comparing two expressions for $Z_{\alpha\beta}$, one obtains

$$\langle\beta|\tilde{q}^{\frac{1}{2}H_c}|\alpha\rangle = \sum_{\lambda\in\mathcal{I}}(n_\lambda)_\alpha{}^\beta\,\chi_\lambda(q)\,. \tag{2.16}$$

The multiplicity $(n_\lambda)_\alpha{}^\beta$ takes values in non-negative integers. In particular, $(n_0)_\alpha{}^\alpha = 1$, since an open string with the same boundary condition at both ends should have the unique vacuum. We therefore obtain

$$\langle\alpha|\tilde{q}^{\frac{1}{2}H_c}|\alpha\rangle = \chi_0(q) + \cdots\,, \tag{2.17}$$

for any boundary state $|\alpha\rangle$. Clearly, the simplest situation is that the right-hand side contains only the vacuum character χ_0. We denote such a state by $|0\rangle$,

$$\langle 0|\tilde{q}^{\frac{1}{2}H_c}|0\rangle = \chi_0(q)\,. \tag{2.18}$$

This state can be realized in the charge-conjugation modular invariant using the Ishibashi states,

$$|0\rangle = \sum_{\lambda\in\mathcal{I}} S_{0\lambda}|(\lambda,\lambda^*)\rangle\!\rangle\,. \tag{2.19}$$

One can check that this state has the desired overlap with itself,

$$\langle 0|\tilde{q}^{\frac{1}{2}H_c}|0\rangle = \sum_\lambda \overline{S_{0\lambda}}S_{0\lambda}\frac{1}{S_{0\lambda}}\chi_\lambda(\tilde{q}) = \sum_\lambda S_{0\lambda}\chi_\lambda(\tilde{q}) = \chi_0(q)\,. \tag{2.20}$$

The remaining states satisfying the boundary condition (2.5) are determined by requiring that their overlap with $|0\rangle$ contain a single character χ_λ,

$$\langle 0|\tilde{q}^{\frac{1}{2}H_c}|\lambda\rangle = \chi_\lambda(q) \quad (\lambda\in\mathcal{I})\,. \tag{2.21}$$

One can solve this equation to obtain $|\lambda\rangle$ in the form

$$|\lambda\rangle = \sum_{\mu\in\mathcal{I}} S_{\lambda\mu}|(\mu,\mu^*)\rangle\!\rangle\,. \tag{2.22}$$

Actually, the overlap of this state with $|0\rangle$ yields χ_λ,

$$\langle 0|\tilde{q}^{\frac{1}{2}H_c}|\lambda\rangle = \sum_\mu \overline{S_{0\mu}}S_{\lambda\mu}\frac{1}{S_{0\mu}}\chi_\mu(\tilde{q}) = \sum_\mu S_{\lambda\mu}\chi_\mu(\tilde{q}) = \chi_\lambda(q)\,. \tag{2.23}$$

Taking the complex conjugation of this equation, we obtain the conjugate representation λ^* of $\lambda\in\mathcal{I}$,

$$\langle\lambda|\tilde{q}^{\frac{1}{2}H_c}|0\rangle = \chi_{\lambda^*}(q)\,. \tag{2.24}$$

One can relate the overlap of two boundary states with the fusion rule coefficients. Suppose that we have two open strings, one of which has the boundary condition $[\lambda^*, 0]$ and the other has $[0,\mu]$ (see Fig.1). The spectrum of these two can be calculated as

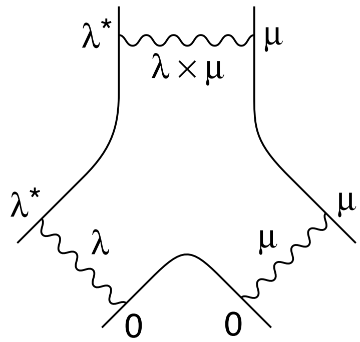

Figure 1: Joining two open strings with boundary condition $[\lambda^*, 0]$ and $[0, \mu]$ yields a string with $[\lambda^*, \mu]$.

$$\langle \lambda^* | \tilde{q}^{\frac{1}{2} H_c} | 0 \rangle = \chi_\lambda(q) \,, \quad \langle 0 | \tilde{q}^{\frac{1}{2} H_c} | \mu \rangle = \chi_\mu(q) \,. \tag{2.25}$$

Since the boundary condition '0' is common to both strings, one can join two strings at the boundary '0' to yield a string with the boundary condition $[\lambda^*, \mu]$. The spectrum of this string can be determined by the fusion of the spectra of the initial strings. In the present case, this is nothing but the fusion of λ and μ. Hence we obtain

$$\langle \lambda^* | \tilde{q}^{\frac{1}{2} H_c} | \mu \rangle = \sum_{\nu \in \mathcal{I}} \mathcal{N}_{\lambda\mu}{}^{\nu} \chi_\nu(q) \,. \tag{2.26}$$

On the other hand, we can explicitly calculate the left-hand side using the Ishibashi states,

$$\begin{aligned} \langle \lambda^* | \tilde{q}^{\frac{1}{2} H_c} | \mu \rangle &= \sum_{\sigma \in \mathcal{I}} \overline{S_{\lambda^* \sigma}} S_{\mu\sigma} \frac{1}{S_{0\sigma}} \chi_\sigma(\tilde{q}) \\ &= \sum_{\sigma \in \mathcal{I}} \overline{S_{\lambda^* \sigma}} S_{\mu\sigma} \frac{1}{S_{0\sigma}} \sum_{\nu \in \mathcal{I}} (S^{-1})_{\sigma\nu} \chi_\nu(q) \\ &= \sum_{\nu, \sigma \in \mathcal{I}} \frac{S_{\lambda\sigma} S_{\mu\sigma} \overline{S_{\nu\sigma}}}{S_{0\sigma}} \chi_\nu(q) \,, \end{aligned} \tag{2.27}$$

where we used the unitarity of S and the property

$$\overline{S_{\lambda\mu}} = S_{\lambda^* \mu} \,. \tag{2.28}$$

Comparison of these two expressions for $\langle \lambda^* | \tilde{q}^{\frac{1}{2} H_c} | \mu \rangle$ yields [1] the Verlinde formula (2.4).

2.2 Twisted boundary states and generalized fusion algebra

We next turn to the case of twisted boundary states and define the generalized fusion algebra [9, 10, 15].

Let G be a group of automorphisms of the chiral algebra $\mathcal{A}$, which is in general a subgroup of the full automorphism group $\mathrm{Aut}(\mathcal{A})$. We restrict ourselves to the case that G is finite

(but not necessarily abelian). We also require that G fixes the Virasoro algebra in $\mathcal{A}$. Then we can modify the boundary condition of the chiral algebra using $\omega \in G$ as follows,

$$(\omega(W_n) - (-1)^h \widetilde{W}_{-n})|\tilde{\alpha}\rangle = 0\,, \tag{2.29}$$

which we call twisted boundary conditions while the condition (2.5) the untwisted one. This boundary condition preserves the conformal symmetry since G fixes the Virasoro algebra by assumption.

An automorphism $\omega \in G$ determines a unitary operator $R(\omega)$ on the representation space $\bigoplus_{\lambda\in\mathcal{I}} \mathcal{H}_\lambda$ of $\mathcal{A}$ through the relation

$$\omega(W_n) = R(\omega) W_n R(\omega)^{-1}\,. \tag{2.30}$$

The operator $R(\omega)$ in general maps the representation space $\mathcal{H}_\lambda$ to the different one, which we denote by $\mathcal{H}_{\omega(\lambda)}$,

$$R(\omega) : \mathcal{H}_\lambda \to \mathcal{H}_{\omega(\lambda)}\,. \tag{2.31}$$

Using this $R(\omega)$, the basis of solutions to (2.29) can be constructed from the Ishibashi states (2.6) for the untwisted condition (2.5) in the following manner,

$$\begin{aligned} |(\lambda,\mu^*);\omega\rangle\!\rangle &= R(\omega)|(\mu,\mu^*)\rangle\!\rangle \\ &= \frac{1}{\sqrt{S_{0\mu}}}\sum_N R(\omega)|\mu;N\rangle \otimes \overline{|\mu;N\rangle} \in \mathcal{H}_\lambda \otimes \widetilde{\mathcal{H}}_{\mu^*} \quad (\lambda = \omega(\mu))\,, \end{aligned} \tag{2.32}$$

where $R(\omega)$ acts only on the holomorphic sector and $\lambda = \omega(\mu)$. One can verify that this state satisfies the twisted boundary condition (2.29) as follows

$$\begin{aligned} (-1)^h \widetilde{W}_{-n}|(\lambda,\mu^*);\omega\rangle\!\rangle &= R(\omega)(-1)^h \widetilde{W}_{-n}|(\mu,\mu^*)\rangle\!\rangle \\ &= R(\omega) W_n |(\mu,\mu^*)\rangle\!\rangle \\ &= R(\omega) W_n R(\omega)^{-1} R(\omega)|(\mu,\mu^*)\rangle\!\rangle \\ &= \omega(W_n)|(\lambda,\mu^*);\omega\rangle\!\rangle\,. \end{aligned} \tag{2.33}$$

We call these states $|(\lambda,\mu^*);\omega\rangle\!\rangle$ twisted Ishibashi states. The untwisted one (2.6) can be considered as the particular case $\omega = 1$ of twisted states. Note that the ω-twisted Ishibashi state $|(\lambda,\mu^*);\omega\rangle\!\rangle$ exists if and only if $\lambda = \omega(\mu)$.

The overlap of twisted Ishibashi states can be calculated in the same way as the untwisted case,

$$\begin{aligned} \langle\!\langle(\lambda,\mu^*);\omega|\tilde{q}^{\frac{1}{2}H_c}|(\lambda',\mu'^*);\omega'\rangle\!\rangle &= \langle\!\langle(\mu,\mu^*)|R(\omega)^\dagger \tilde{q}^{\frac{1}{2}H_c} R(\omega')|(\mu',\mu'^*)\rangle\!\rangle \\ &= \delta_{\mu\mu'}\frac{1}{S_{0\mu}}\sum_N \langle\mu;N|R(\omega^{-1}\omega')\tilde{q}^{L_0-\frac{c}{24}}|\mu;N\rangle\,. \end{aligned} \tag{2.34}$$

Clearly, this expression vanishes unless $\omega^{-1}\omega'(\mu) = \mu$, or equivalently $\omega^{-1}\omega' \in \mathcal{S}(\mu)$, where $\mathcal{S}(\mu) \subset G$ is the stabilizer of $\mu \in \mathcal{I}$,

$$\mathcal{S}(\mu) = \{\omega \in G \,|\, \omega(\mu) = \mu\}\,. \tag{2.35}$$

Using the relations $\lambda = \omega(\mu)$ and $\lambda' = \omega'(\mu') = \omega'(\mu)$, we can rewrite the condition $\omega^{-1}\omega' \in \mathcal{S}(\mu)$ into that for λ and λ',

$$\lambda' = \omega'(\mu) = \omega\omega^{-1}\omega'(\mu) = \omega(\mu) = \lambda\,. \tag{2.36}$$

Conversely, $\lambda = \lambda'$ implies $\omega^{-1}\omega' \in \mathcal{S}(\mu)$,

$$\omega^{-1}\omega'(\mu) = \omega^{-1}(\lambda') = \omega^{-1}(\lambda) = \mu\,. \tag{2.37}$$

Putting these things together, we can express the overlap of twisted Ishibashi states in the following form

$$\langle\!\langle(\lambda,\mu^*);\omega|\tilde{q}^{\frac{1}{2}H_c}|(\lambda',\mu'^*);\omega'\rangle\!\rangle = \delta_{(\lambda,\mu^*)(\lambda',\mu'^*)}\frac{1}{S_{0\mu}}\chi_\mu^{\omega^{-1}\omega'}(\tilde{q}) \quad (\lambda=\omega(\mu),\,\lambda'=\omega'(\mu'))\,, \tag{2.38}$$

where χ_μ^ω $(\omega \in \mathcal{S}(\mu))$ is the twining character [24] of $\mathcal{A}$,

$$\chi_\mu^\omega(q) = \mathrm{Tr}_{\mathcal{H}_\mu} R(\omega) q^{L_0 - \frac{c}{24}}\,. \tag{2.39}$$

As we have noted above, χ_μ^ω for a given $\mu \in \mathcal{I}$ vanishes unless $\omega \in \mathcal{S}(\mu)$. Equivalently, for a given $\omega \in G$, χ_μ^ω exists if and only if $\omega(\mu) = \mu$. To express the condition for non-vanishing χ_λ^ω for a given $\omega \in G$, we introduce a set $\mathcal{I}(\omega)$ which consists of representations of $\mathcal{A}$ fixed by ω

$$\mathcal{I}(\omega) = \{\lambda \in \mathcal{I}\,|\,\omega(\lambda) = \lambda\}\,. \tag{2.40}$$

It is useful to note that the following two conditions are equivalent,

$$\lambda \in \mathcal{I}(\omega) \;\Leftrightarrow\; \omega \in \mathcal{S}(\lambda)\,. \tag{2.41}$$

Since the twisted Ishibashi state $|(\lambda,\mu^*);\omega\rangle\!\rangle$ composed from the representation (λ,μ^*) exists only when $\lambda = \omega(\mu)$, the set of ω-twisted Ishibashi states available in the charge-conjugation invariant (2.11) have the form

$$\{|(\lambda,\lambda^*);\omega\rangle\!\rangle\,|\,\lambda \in \mathcal{I}(\omega)\}\,. \tag{2.42}$$

A general boundary state $|\tilde{\alpha}\rangle$ satisfying the boundary condition (2.29) in the charge-conjugation modular invariant is therefore expressed as follows,

$$|\tilde{\alpha}\rangle = \sum_{\lambda\in\mathcal{I}(\omega)} \Psi^\omega_{\tilde{\alpha}(\lambda,\lambda^*)}|(\lambda,\lambda^*);\omega\rangle\!\rangle\,. \tag{2.43}$$

In order to determine the coefficients $\Psi^\omega_{\tilde{\alpha}(\lambda,\lambda^*)}$, we consider the cylinder amplitude $\langle 0|\tilde{q}^{\frac{1}{2}H_c}|\tilde{\alpha}\rangle$ with boundary conditions $0 \in \mathcal{I}$ and $\tilde{\alpha}$. Since the boundary condition of the corresponding open string is twisted by ω at only one end (see Fig.2), the chiral algebra in the open-string channel is the twisted chiral algebra $\mathcal{A}^\omega$ of $\mathcal{A}$ associated with $\omega \in G$, which is generated by the currents of $\mathcal{A}$ with the twisted boundary condition,

$$W(e^{2\pi i}z) = \omega(W(z))\,. \tag{2.44}$$

The cylinder amplitude $\langle 0|\tilde{q}^{\frac{1}{2}H_c}|\tilde{\alpha}\rangle$ is therefore expanded into the characters of $\mathcal{A}^\omega$.

We denote by $\mathcal{I}^\omega$ the set of all the irreducible representations of the twisted chiral algebra $\mathcal{A}^\omega$ and by $\chi_{\tilde{\lambda}}$ the character of $\tilde{\lambda} \in \mathcal{I}^\omega$. The modular transformation of $\chi_{\tilde{\lambda}}$ can be expanded into the twining characters (2.39) of $\mathcal{A}$. Actually, evaluating the cylinder amplitude $\langle 0|\tilde{q}^{\frac{1}{2}H_c}|\tilde{\alpha}\rangle$ in the closed-string channel yields a linear combination of the overlap $\langle\!\langle(\lambda,\lambda^*)|\tilde{q}^{\frac{1}{2}H_c}|(\lambda,\lambda^*);\omega\rangle\!\rangle$, which in turn provides the twining character $\chi^\omega_\lambda(\tilde{q})$ as we have shown in (2.38). [4] Accordingly, we can express the modular transformation of the character $\chi_{\tilde{\lambda}}$ in the following form,

$$\chi_{\tilde{\lambda}}(q) = \sum_{\mu\in\mathcal{I}(\omega)} S^\omega_{\tilde{\lambda}\mu}\chi^\omega_\mu(\tilde{q}) \quad (\tilde{\lambda}\in\mathcal{I}^\omega)\,. \tag{2.45}$$

The matrix S^ω relates the representations of $\mathcal{A}^\omega$ with those of $\mathcal{A}$ fixed by ω. We assume that S^ω is unitary. In particular, two sets, $\mathcal{I}^\omega$ and $\mathcal{I}(\omega)$, have the same order, $|\mathcal{I}^\omega| = |\mathcal{I}(\omega)|$. This holds for many examples including the case of an affine Lie algebra of untwisted type with the diagram automorphism of its horizontal Lie algebra as an automorphism ω. [5]

We can determine the coefficients of twisted Ishibashi states in a way parallel to the untwisted one, namely we require the condition

$$\langle 0|\tilde{q}^{\frac{1}{2}H_c}|\tilde{\lambda}\rangle = \chi_{\tilde{\lambda}}(q) \quad (\tilde{\lambda}\in\mathcal{I}^\omega)\,. \tag{2.46}$$

In the same way as the untwisted case, twisted boundary states satisfying the above condition can be constructed using the modular transformation matrix of the chiral algebra in the open-string channel, this time $\mathcal{A}^\omega$,

$$|\tilde{\lambda}\rangle = \sum_{\mu\in\mathcal{I}(\omega)} S^\omega_{\tilde{\lambda}\mu}|(\mu,\mu^*);\omega\rangle\!\rangle\,. \tag{2.47}$$

We can check this expression yields the desired overlap (2.46) with $|0\rangle$,

$$\langle 0|\tilde{q}^{\frac{1}{2}H_c}|\tilde{\lambda}\rangle = \sum_{\mu\in\mathcal{I}(\omega)} \overline{S_{0\mu}}S^\omega_{\tilde{\lambda}\mu}\frac{1}{S_{0\mu}}\chi^\omega_\mu(\tilde{q}) = \sum_{\mu\in\mathcal{I}(\omega)} S^\omega_{\tilde{\lambda}\mu}\chi^\omega_\mu(\tilde{q}) = \chi_{\tilde{\lambda}}(q)\,. \tag{2.48}$$

Having obtained the explicit form of the twisted states, one can calculate the overlap of the twisted state $|\tilde{\mu}\rangle$ ($\tilde{\mu}\in\mathcal{I}^\omega$) with generic untwisted states,

$$\begin{aligned}\langle\lambda^*|\tilde{q}^{\frac{1}{2}H_c}|\tilde{\mu}\rangle &= \sum_{\sigma\in\mathcal{I}(\omega)} \overline{S_{\lambda^*\sigma}}S^\omega_{\tilde{\mu}\sigma}\frac{1}{S_{0\sigma}}\chi^\omega_\sigma(\tilde{q}) \\ &= \sum_{\sigma\in\mathcal{I}(\omega)} \overline{S_{\lambda^*\sigma}}S^\omega_{\tilde{\mu}\sigma}\frac{1}{S_{0\sigma}}\sum_{\tilde{\nu}\in\mathcal{I}^\omega}(S^\omega)^{-1}_{\sigma\tilde{\nu}}\chi_{\tilde{\nu}}(q) \\ &= \sum_{\tilde{\nu}\in\mathcal{I}^\omega,\,\sigma\in\mathcal{I}(\omega)} \frac{S_{\lambda\sigma}S^\omega_{\tilde{\mu}\sigma}\overline{S^\omega_{\tilde{\nu}\sigma}}}{S_{0\sigma}}\chi_{\tilde{\nu}}(q)\,.\end{aligned} \tag{2.49}$$

[4] This argument gives an explanation for the fact [24] that the twining character for an affine Lie algebra $\mathfrak{g}^{(1)}$ with the diagram automorphism of its horizontal Lie algebra as an automorphism ω of order r can be expressed by characters of a twisted affine Lie algebra $\mathfrak{g}'^{(r)}$, since the modular transformation of characters of a twisted affine Lie algebra $\mathfrak{g}^{(r)}$ is expanded into characters of a (generally different) twisted affine Lie algebra $\mathfrak{g}'^{(r)}$ [32].

[5] A detailed study of the matrix S^ω for the case of $\omega^2 = 1$ is given in [9].

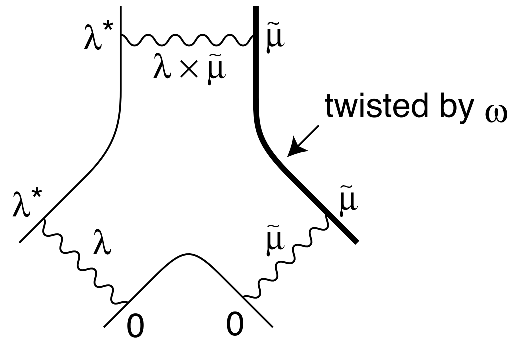

Figure 2: Joining two open strings with boundary condition $[\lambda^*, 0]$ and $[0, \tilde{\mu}]$ yields a string with $[\lambda^*, \tilde{\mu}]$. The thick line stands for a twist by the automorphism $\omega \in G$.

In the same way as the case of untwisted states, one can regard this as the fusion of two open strings, $[\lambda^*, 0]$ and $[0, \tilde{\mu}]$ (see Fig.2). The spectrum of these two strings contains only one representation, λ and $\tilde{\mu}$, respectively. Therefore the spectrum of the joined string $[\lambda^*, \tilde{\mu}]$ can be considered to express the fusion $(\lambda) \times (\tilde{\mu})$ of two representations, $\lambda \in \mathcal{I}$ and $\tilde{\mu} \in \mathcal{I}^\omega$. Since the string $[\lambda^*, \tilde{\mu}]$ has a twist by ω at only one end $\tilde{\mu}$, its spectrum consists of representations of the twisted chiral algebra $\mathcal{A}^\omega$. Hence one can conclude that the product $(\lambda) \times (\tilde{\mu})$ is expanded into representations of $\mathcal{A}^\omega$,

$$(\lambda) \times (\tilde{\mu}) = \sum_{\tilde{\nu} \in \mathcal{I}^\omega} \mathcal{N}_{\lambda\tilde{\mu}}{}^{\tilde{\nu}} (\tilde{\nu}) \qquad (\lambda \in \mathcal{I},\ \tilde{\mu} \in \mathcal{I}^\omega)\,. \tag{2.50}$$

The coefficients $\mathcal{N}_{\lambda\tilde{\mu}}{}^{\tilde{\nu}}$ express the multiplicity of representation $(\tilde{\nu})$ in the product $(\lambda) \times (\tilde{\mu})$ and take values in non-negative integers. In terms of these coefficients, the overlap of untwisted states with twisted states can be written in the following form,

$$\langle \lambda^* | \tilde{q}^{\frac{1}{2} H_c} | \tilde{\mu} \rangle = \sum_{\tilde{\nu} \in \mathcal{I}^\omega} \mathcal{N}_{\lambda\tilde{\mu}}{}^{\tilde{\nu}} \chi_{\tilde{\nu}}(q)\,. \tag{2.51}$$

Comparing this with the closed-string channel calculation (2.49), one can obtain a formula for $\mathcal{N}_{\lambda\tilde{\mu}}{}^{\tilde{\nu}}$,

$$\mathcal{N}_{\lambda\tilde{\mu}}{}^{\tilde{\nu}} = \sum_{\rho \in \mathcal{I}(\omega)} \frac{S_{\lambda\rho} S^\omega_{\tilde{\mu}\rho} \overline{S^\omega_{\tilde{\nu}\rho}}}{S_{0\rho}} \qquad (\lambda \in \mathcal{I};\ \tilde{\mu}, \tilde{\nu} \in \mathcal{I}^\omega)\,. \tag{2.52}$$

The above result can be readily generalized to the fusion of two generic representations. To state the result in a concise form, we treat the untwisted states as the particular case $\omega = 1$ of the twisted states, and introduce a set $\hat{\mathcal{I}}$ of all the representations of twisted chiral algebras $\mathcal{A}^\omega$ for $\omega \in G$,

$$\hat{\mathcal{I}} = \coprod_{\omega \in G} \mathcal{I}^\omega\,, \tag{2.53}$$

which contains $\mathcal{I} = \mathcal{I}^{\omega=1}$ as a subset. We use a capital letter such as L for expressing an element of $\hat{\mathcal{I}}$ and denote the automorphism type of $L \in \hat{\mathcal{I}}$ by putting a subscript like $\omega_L \in G$.

In this notation, the modular transformation, eqs.(2.2) and (2.45), of characters of $\mathcal{A}^\omega$ can be written as follows,

$$\chi_L(q) = \sum_{\lambda \in \mathcal{I}(\omega_L)} S^{\omega_L}_{L\lambda}\, \chi^{\omega_L}_\lambda(\tilde{q}) \quad (L \in \hat{\mathcal{I}})\,. \tag{2.54}$$

If L is a representation of $\mathcal{A}$, $L \in \mathcal{I}$, the modular transformation matrix S^{ω_L} is the ordinary S-matrix and the twining character $\chi^{\omega_L}_\lambda$ is nothing but the character χ_λ of $\mathcal{A}$. The boundary states are labeled by the element of $\hat{\mathcal{I}}$ and take the form given in eqs.(2.22) and (2.47),

$$|L\rangle = \sum_{\lambda \in \mathcal{I}(\omega_L)} S^{\omega_L}_{L\lambda} |(\lambda, \lambda^*); \omega_L\rangle\!\rangle \quad (L \in \hat{\mathcal{I}})\,. \tag{2.55}$$

The overlap of $|L\rangle$ with the untwisted state $|0\rangle$ yields the character χ_L in the open-string channel,

$$\langle 0 | \tilde{q}^{\frac{1}{2}H_c} | L\rangle = \chi_L(q) \quad (L \in \hat{\mathcal{I}})\,, \tag{2.56}$$

which means that the open string $[0, L]$ has a spectrum consisting of only one representation $L \in \hat{\mathcal{I}}$.

Taking the complex conjugation of this equation, one obtains the spectrum of the string $[L, 0]$. Since the string $[L, 0]$ has an orientation opposite to that of $[0, L]$, its chiral algebra has a boundary condition twisted by ω_L^{-1} instead of ω_L. The spectrum of the string $[L, 0]$ is therefore expanded into characters of $\mathcal{A}^{\omega_L^{-1}}$. Since the string $[0, L]$ has only one representation, the spectrum of the string $[L, 0]$ also consists of only one representation. We call this the conjugate representation of L and denote it by $L^* \in \mathcal{I}^{\omega_L^{-1}} \subset \hat{\mathcal{I}}$,

$$\langle L | \tilde{q}^{\frac{1}{2}H_c} | 0\rangle = \chi_{L^*}(q) \quad (L \in \hat{\mathcal{I}})\,. \tag{2.57}$$

This generalizes the complex conjugation of ordinary representations (see (2.24)). Calculating the left-hand side of eq.(2.57) in terms of the Ishibashi states yields

$$\chi_{L^*}(q) = \sum_{\lambda \in \mathcal{I}(\omega_L)} \overline{S^{\omega_L}_{L\lambda}}\, \chi^{\omega_L^{-1}}_\lambda(\tilde{q})\,. \tag{2.58}$$

Comparing this with eq.(2.54), we obtain the following formula

$$\overline{S^{\omega_L}_{L\lambda}} = S^{\omega_L^{-1}}_{L^*\lambda} \quad (L \in \hat{\mathcal{I}})\,, \tag{2.59}$$

which reduces to eq.(2.28) for the case of $\omega_L = 1$.

Since the spectrum of two open strings, $[L^*, 0]$ and $[0, M]$, consists of only one representation, L and M, respectively, the spectrum of the joined string $[L^*, M]$ is given by the fusion of two representations L and M. From the expression given in eq.(2.55), one can calculate

the overlap of $|L^*\rangle$ and $|M\rangle$ as follows,

$$\begin{aligned}
\langle L^*|\tilde{q}^{\frac{1}{2}H_c}|M\rangle &= \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)} \overline{S^{\omega_L^{-1}}_{L^*\lambda}} S^{\omega_M}_{M\lambda}\, \langle\!\langle(\lambda,\lambda^*);\omega_L^{-1}|\tilde{q}^{\frac{1}{2}H_c}|(\lambda,\lambda^*);\omega_M\rangle\!\rangle \\
&= \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)} S^{\omega_L}_{L\lambda} S^{\omega_M}_{M\lambda} \frac{1}{S_{0\lambda}} \chi^{\omega_L\omega_M}_{\lambda}(\tilde{q}) \\
&= \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)} S^{\omega_L}_{L\lambda} S^{\omega_M}_{M\lambda} \frac{1}{S_{0\lambda}} \sum_{N\in\mathcal{I}^{\omega_L\omega_M}} (S^{\omega_L\omega_M})^{-1}_{\lambda N}\, \chi_N(q) \\
&= \sum_{N\in\mathcal{I}^{\omega_L\omega_M}} \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)} \frac{S^{\omega_L}_{L\lambda} S^{\omega_M}_{M\lambda} \overline{S^{\omega_L\omega_M}_{N\lambda}}}{S_{0\lambda}} \chi_N(q)\,,
\end{aligned} \tag{2.60}$$

where we used the formula (2.38) and the unitarity of $S^{\omega_L\omega_M}$. This equation shows that the fusion of two representations L and M ($L, M \in \hat{\mathcal{I}}$) can be expanded into representations of the automorphism type $\omega_L\omega_M \in G$, which are also the elements of $\hat{\mathcal{I}}$. Therefore the set $\hat{\mathcal{I}}$ is closed under the fusion of representations and forms an algebra

$$(L) \times (M) = \sum_{N\in\hat{\mathcal{I}}} \widehat{\mathcal{N}}_{LM}{}^{N} (N)\,, \tag{2.61}$$

which generalizes the fusion algebra of ordinary untwisted representations. We call this the generalized fusion algebra and denote it by $\mathcal{F}(\mathcal{A};G)$. [6] The coefficients $\widehat{\mathcal{N}}_{LM}{}^{N}$ take values in non-negative integers and we call them the generalized fusion coefficients. As is shown in the overlap (2.60), the generalized fusion coefficients are given by the following formula

$$\widehat{\mathcal{N}}_{LM}{}^{N} = \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)} \frac{S^{\omega_L}_{L\lambda} S^{\omega_M}_{M\lambda} \overline{S^{\omega_N}_{N\lambda}}}{S_{0\lambda}} \times \delta_{\omega_L\omega_M,\omega_N}\,, \tag{2.62}$$

which generalizes the Verlinde formula (2.4) for the ordinary fusion coefficients.

As we have argued above, the fusion of a representation L with M has the automorphism type $\omega_L\omega_M$ and is given by a linear combination of the elements in $\mathcal{I}^{\omega_L\omega_M}$. An immediate consequence of this fact is that $(L)\times(M) \neq (M)\times(L)$ in general, since two sets $\mathcal{I}^{\omega_L\omega_M}$ and $\mathcal{I}^{\omega_M\omega_L}$ are distinct from each other if $\omega_L\omega_M \neq \omega_M\omega_L$. Therefore the generalized fusion algebra $\mathcal{F}(\mathcal{A};G)$ is *non-commutative* if the automorphism group G is non-abelian. In contrast to this, there seems to be no special reason for a generalized fusion algebra to be non-commutative for an abelian G. Actually, it follows from the definition (2.62) of the generalized fusion coefficients that $(L)\times(M) = (M)\times(L)$ if $\omega_L\omega_M = \omega_M\omega_L$. In particular, $\mathcal{F}(\mathcal{A};G)$ is commutative if G is abelian. The ordinary fusion algebra $\mathcal{F}(\mathcal{A})$ corresponds to the trivial case $\mathcal{F}(\mathcal{A};\{1\})$ and is hence commutative.

2.3 An example: generalized fusion algebras for $so(8)_1$

As an illustration of our argument, we give the explicit form of the generalized fusion algebra for the affine Lie algebra $so(8)_1$ and its triality automorphism group, which is iso-

[6] $\mathcal{F}$ stands for fusion.

morphic to the symmetric group S_3. Since S_3 is non-abelian, the generalized fusion algebra $\mathcal{F}(so(8)_1; S_3)$ [7] is non-commutative.

There are four irreducible representations for $so(8)_1$,

$$\mathcal{I} = \{O = \Lambda_0, V = \Lambda_1, S = \Lambda_3, C = \Lambda_4\}, \tag{2.63}$$

where Λ_i is the fundamental weight of $so(8)_1$ and each symbol stands for the vacuum, vector, spinor and conjugate spinor representation, respectively. The modular transformation matrix reads

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{2.64}$$

where the rows and the columns are ordered as (2.63).

The chiral algebra $so(8)_1$ has an automorphism group consisting of all the permutations of three legs of the Dynkin diagram for the horizontal subalgebra $so(8) \subset so(8)_1$, which is isomorphic to the symmetric group S_3. The automorphism group S_3 is generated by two elements π and σ, which act on the elements of $\mathcal{I}$ as

$$\begin{aligned} &\pi : V \mapsto S \mapsto C \mapsto V\,,\ O \mapsto O\,, \\ &\sigma : O \mapsto O\,,\ V \mapsto V\,,\ S \mapsto C \mapsto S\,. \end{aligned} \tag{2.65}$$

In terms of π and σ, the elements of S_3 can be expressed as follows,

$$S_3 = \langle \pi, \sigma \rangle = \{1, \pi, \pi^2, \sigma, \pi\sigma, \pi^2\sigma\}\,. \tag{2.66}$$

In order to obtain the generalized fusion algebra, we need to know twisted representations and their modular transformation for each element of S_3. For π and π^2, the fixed point set of the automorphism consists of a single element

$$\mathcal{I}(\pi) = \mathcal{I}(\pi^2) = \{O\}\,. \tag{2.67}$$

Correspondingly, there is only one representation for each twisted chiral algebra, which is isomorphic to the twisted affine Lie algebra $D_4^{(3)}$ at level 1. (The irreducible representations and their modular transformation matrix for the affine Lie algebra $D_4^{(r)}$ ($r = 1, 2, 3$) are presented, *e.g.*, in [25].) We denote the twisted representations for π and π^2 as follows,

$$\mathcal{I}^{\pi} = \{(0)_{\pi}\}\,, \quad \mathcal{I}^{\pi^2} = \{(0)_{\pi^2}\}\,. \tag{2.68}$$

The modular transformation matrix in this case is simply a number

$$S^{\pi} = (1)\,, \quad S^{\pi^2} = (1)\,. \tag{2.69}$$

There are two representations fixed by $\sigma \in S_3$,

$$\mathcal{I}(\sigma) = \{O,\, V\}\,. \tag{2.70}$$

[7] For brevity, we denote the automorphism group itself by S_3.

The twisted chiral algebra for this case is isomorphic to $D_4^{(2)}$ at level 1. There are two representations of the twisted chiral algebra for σ, which we denote as

$$\mathcal{I}^\sigma = \{(0)_\sigma,\, (1)_\sigma\}\,. \tag{2.71}$$

The modular transformation matrix reads

$$S^\sigma = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\,. \tag{2.72}$$

For the other elements $\pi\sigma$ and $\pi^2\sigma$, one can proceed in the same way as σ to obtain

$$\mathcal{I}(\pi\sigma) = \{O,\, C\}\,, \qquad \mathcal{I}^{\pi\sigma} = \{(0)_{\pi\sigma},\, (1)_{\pi\sigma}\}\,, \tag{2.73}$$

$$\mathcal{I}(\pi^2\sigma) = \{O,\, S\}\,, \qquad \mathcal{I}^{\pi^2\sigma} = \{(0)_{\pi^2\sigma},\, (1)_{\pi^2\sigma}\}\,. \tag{2.74}$$

The modular transformation matrix is the same as that for σ,

$$S^{\pi\sigma} = S^{\pi^2\sigma} = S^\sigma\,. \tag{2.75}$$

Consequently, the set $\hat{\mathcal{I}}$ of all the twisted representations consists of 12 elements,

$$|\hat{\mathcal{I}}| = |\mathcal{I}| + |\mathcal{I}^\pi| + |\mathcal{I}^{\pi^2}| + |\mathcal{I}^\sigma| + |\mathcal{I}^{\pi\sigma}| + |\mathcal{I}^{\pi^2\sigma}| = 4+1+1+2+2+2 = 12\,. \tag{2.76}$$

Since all the modular transformation matrices are real, the relation (2.59) implies that the representations in $\mathcal{I}, \mathcal{I}^\sigma, \mathcal{I}^{\pi\sigma}$ and $\mathcal{I}^{\pi^2\sigma}$ are self-conjugate. On the other hand, for the representations in $\mathcal{I}^\pi$ and $\mathcal{I}^{\pi^2}$, the conjugation changes the automorphism type, $(0)_\pi^* = (0)_{\pi^2}$, since $\pi^{-1} = \pi^2$.

Having obtained twisted representations and their modular transformation matrices, it is straightforward to calculate the generalized fusion coefficients of $\mathcal{F}(so(8)_1; S_3)$ using the formula (2.62). For example, the coefficient $\widehat{\mathcal{N}}_{(0)_\pi (0)_\sigma}{}^{(0)_{\pi\sigma}}$ can be obtained as follows,

$$\widehat{\mathcal{N}}_{(0)_\pi (0)_\sigma}{}^{(0)_{\pi\sigma}} = \sum_{\lambda \in \mathcal{I}(\pi)\cap\mathcal{I}(\sigma)} \frac{S^\pi_{(0)_\pi\,\lambda}\, S^\sigma_{(0)_\sigma\,\lambda}\, \overline{S^{\pi\sigma}_{(0)_{\pi\sigma}\,\lambda}}}{S_{O\lambda}} = \frac{S^\pi_{(0)_\pi\,O}\, S^\sigma_{(0)_\sigma\,O}\, \overline{S^{\pi\sigma}_{(0)_{\pi\sigma}\,O}}}{S_{OO}} = 1\,. \tag{2.77}$$

The other cases can be calculated in the same manner; we give the result in Table 1. As this table shows explicitly, the generalized fusion algebra $\mathcal{F}(so(8)_1; S_3)$ is non-commutative. For instance, the fusion of $(0)_\pi$ with $(0)_\sigma$ yields

$$\begin{aligned} (0)_\pi \times (0)_\sigma &= (0)_{\pi\sigma} + (1)_{\pi\sigma}\,, \\ (0)_\sigma \times (0)_\pi &= (0)_{\pi^2\sigma} + (1)_{\pi^2\sigma}\,. \end{aligned} \tag{2.78}$$

Since $\pi^2\sigma \neq \pi\sigma$, these two expressions are distinct from each other, namely $(0)_\sigma \times (0)_\pi \neq (0)_\pi \times (0)_\sigma$, which shows the non-commutativity of this algebra. One can also check the associativity of this algebra, for example,

$$\begin{aligned} (0)_\pi \times \big((0)_\pi \times (0)_\sigma\big) &= (0)_\pi \times (0)_{\pi\sigma} + (0)_\pi \times (1)_{\pi\sigma} = 2\,(0)_{\pi^2\sigma} + 2\,(1)_{\pi^2\sigma}\,, \\ \big((0)_\pi \times (0)_\pi\big) \times (0)_\sigma &= 2\,(0)_{\pi^2} \times (0)_\sigma = 2\,(0)_{\pi^2\sigma} + 2\,(1)_{\pi^2\sigma}\,. \end{aligned} \tag{2.79}$$

	O	V	S	C	$(0)_\pi$	$(0)_{\pi^2}$
O	O	V	S	C	$(0)_\pi$	$(0)_{\pi^2}$
V	V	O	C	S	$(0)_\pi$	$(0)_{\pi^2}$
S	S	C	O	V	$(0)_\pi$	$(0)_{\pi^2}$
C	C	S	V	O	$(0)_\pi$	$(0)_{\pi^2}$
$(0)_\pi$	$(0)_\pi$	$(0)_\pi$	$(0)_\pi$	$(0)_\pi$	$2\,(0)_{\pi^2}$	$O+V+S+C$
$(0)_{\pi^2}$	$(0)_{\pi^2}$	$(0)_{\pi^2}$	$(0)_{\pi^2}$	$(0)_{\pi^2}$	$O+V+S+C$	$2\,(0)_\pi$
$(0)_\sigma$	$(0)_\sigma$	$(0)_\sigma$	$(1)_\sigma$	$(1)_\sigma$	$(0)_{\pi^2\sigma}+(1)_{\pi^2\sigma}$	$(0)_{\pi\sigma}+(1)_{\pi\sigma}$
$(1)_\sigma$	$(1)_\sigma$	$(1)_\sigma$	$(0)_\sigma$	$(0)_\sigma$	$(0)_{\pi^2\sigma}+(1)_{\pi^2\sigma}$	$(0)_{\pi\sigma}+(1)_{\pi\sigma}$
$(0)_{\pi\sigma}$	$(0)_{\pi\sigma}$	$(1)_{\pi\sigma}$	$(1)_{\pi\sigma}$	$(0)_{\pi\sigma}$	$(0)_\sigma+(1)_\sigma$	$(0)_{\pi^2\sigma}+(1)_{\pi^2\sigma}$
$(1)_{\pi\sigma}$	$(1)_{\pi\sigma}$	$(0)_{\pi\sigma}$	$(0)_{\pi\sigma}$	$(1)_{\pi\sigma}$	$(0)_\sigma+(1)_\sigma$	$(0)_{\pi^2\sigma}+(1)_{\pi^2\sigma}$
$(0)_{\pi^2\sigma}$	$(0)_{\pi^2\sigma}$	$(1)_{\pi^2\sigma}$	$(0)_{\pi^2\sigma}$	$(1)_{\pi^2\sigma}$	$(0)_{\pi\sigma}+(1)_{\pi\sigma}$	$(0)_\sigma+(1)_\sigma$
$(1)_{\pi^2\sigma}$	$(1)_{\pi^2\sigma}$	$(0)_{\pi^2\sigma}$	$(1)_{\pi^2\sigma}$	$(0)_{\pi^2\sigma}$	$(0)_{\pi\sigma}+(1)_{\pi\sigma}$	$(0)_\sigma+(1)_\sigma$

	$(0)_\sigma$	$(1)_\sigma$	$(0)_{\pi\sigma}$	$(1)_{\pi\sigma}$	$(0)_{\pi^2\sigma}$	$(1)_{\pi^2\sigma}$
O	$(0)_\sigma$	$(1)_\sigma$	$(0)_{\pi\sigma}$	$(1)_{\pi\sigma}$	$(0)_{\pi^2\sigma}$	$(1)_{\pi^2\sigma}$
V	$(0)_\sigma$	$(1)_\sigma$	$(1)_{\pi\sigma}$	$(0)_{\pi\sigma}$	$(1)_{\pi^2\sigma}$	$(0)_{\pi^2\sigma}$
S	$(1)_\sigma$	$(0)_\sigma$	$(1)_{\pi\sigma}$	$(0)_{\pi\sigma}$	$(0)_{\pi^2\sigma}$	$(1)_{\pi^2\sigma}$
C	$(1)_\sigma$	$(0)_\sigma$	$(0)_{\pi\sigma}$	$(1)_{\pi\sigma}$	$(1)_{\pi^2\sigma}$	$(0)_{\pi^2\sigma}$
$(0)_\pi$	$(0)_{\pi\sigma}+(1)_{\pi\sigma}$	$(0)_{\pi\sigma}+(1)_{\pi\sigma}$	$(0)_{\pi^2\sigma}+(1)_{\pi^2\sigma}$	$(0)_{\pi^2\sigma}+(1)_{\pi^2\sigma}$	$(0)_\sigma+(1)_\sigma$	$(0)_\sigma+(1)_\sigma$
$(0)_{\pi^2}$	$(0)_{\pi^2\sigma}+(1)_{\pi^2\sigma}$	$(0)_{\pi^2\sigma}+(1)_{\pi^2\sigma}$	$(0)_\sigma+(1)_\sigma$	$(0)_\sigma+(1)_\sigma$	$(0)_{\pi\sigma}+(1)_{\pi\sigma}$	$(0)_{\pi\sigma}+(1)_{\pi\sigma}$
$(0)_\sigma$	$O+V$	$S+C$	$(0)_{\pi^2}$	$(0)_{\pi^2}$	$(0)_\pi$	$(0)_\pi$
$(1)_\sigma$	$S+C$	$O+V$	$(0)_{\pi^2}$	$(0)_{\pi^2}$	$(0)_\pi$	$(0)_\pi$
$(0)_{\pi\sigma}$	$(0)_\pi$	$(0)_\pi$	$O+C$	$V+S$	$(0)_{\pi^2}$	$(0)_{\pi^2}$
$(1)_{\pi\sigma}$	$(0)_\pi$	$(0)_\pi$	$V+S$	$O+C$	$(0)_{\pi^2}$	$(0)_{\pi^2}$
$(0)_{\pi^2\sigma}$	$(0)_{\pi^2}$	$(0)_{\pi^2}$	$(0)_\pi$	$(0)_\pi$	$O+S$	$V+C$
$(1)_{\pi^2\sigma}$	$(0)_{\pi^2}$	$(0)_{\pi^2}$	$(0)_\pi$	$(0)_\pi$	$V+C$	$O+S$

Table 1: Multiplication table of the generalized fusion algebra $\mathcal{F}(so(8)_1; S_3)$. The subscripts stand for the automorphism type of twisted representations. Since $\sigma\pi = \pi^2\sigma \neq \pi\sigma$, this algebra is non-commutative.

A proof for the associativity of the generalized fusion algebras is given in the next section.

The generalized fusion algebra $\mathcal{F}(so(8)_1; S_3)$ contains non-trivial subalgebras other than the ordinary fusion algebra $\mathcal{F}(so(8)_1)$. Actually, a subalgebra of $\mathcal{F}(so(8)_1; S_3)$ exists for each subgroup of S_3,

$$\{1\},\ \langle\sigma\rangle,\ \langle\pi\sigma\rangle,\ \langle\pi^2\sigma\rangle,\ \langle\pi\rangle = A_3 \cong \mathbb{Z}_3\,. \tag{2.80}$$

For example, corresponding to the three elements $\{1, \pi, \pi^2\}$ of the alternating group $A_3 \subset S_3$, the representations in $\mathcal{I}, \mathcal{I}^\pi$ and $\mathcal{I}^{\pi^2}$ form a subalgebra which we denote as $\mathcal{F}(so(8)_1; A_3)$. The generalized fusion algebra $\mathcal{F}(so(8)_1; S_3)$ is then decomposed into representations of $\mathcal{F}(so(8)_1; A_3)$. Indeed, from Table 1, one can check that the set $\mathcal{I}^\sigma \amalg \mathcal{I}^{\pi\sigma} \amalg \mathcal{I}^{\pi^2\sigma}$ is invariant under the action of $\mathcal{F}(so(8)_1; A_3)$ and hence gives a representation of it. Together with the regular representation of $\mathcal{F}(so(8)_1; A_3)$ on $\mathcal{I} \amalg \mathcal{I}^\pi \amalg \mathcal{I}^{\pi^2}$, we eventually obtain two representations of $\mathcal{F}(so(8)_1; A_3)$ from the decomposition of the regular representation of $\mathcal{F}(so(8)_1; S_3)$.

3 Irreducible representations

For ordinary fusion algebras, the formula (2.62) for the generalized fusion coefficients reduces to the ordinary Verlinde formula

$$\mathcal{N}_{LM}{}^{N} = \sum_{\lambda\in\mathcal{I}} \frac{S_{L\lambda} S_{M\lambda} \overline{S_{N\lambda}}}{S_{0\lambda}}\,, \tag{3.1}$$

from which one can show that the generalized quantum dimension

$$b_\lambda(L) = \frac{S_{L\lambda}}{S_{0\lambda}} \quad (\lambda \in \mathcal{I}) \tag{3.2}$$

realizes a one-dimensional representation of the fusion algebra $\mathcal{F}(\mathcal{A})$,

$$b_\lambda(L) b_\lambda(M) = \sum_{N\in\mathcal{I}} \mathcal{N}_{LM}{}^{N}\, b_\lambda(N)\,. \tag{3.3}$$

The aim of this section is to extend this result to the case of the generalized fusion algebra $\mathcal{F}(\mathcal{A}; G)$ for a general finite group G. As we shall show below, we have irreducible representations with dimension greater than 1 due to the non-commutativity of $\mathcal{F}(\mathcal{A}; G)$ if G is non-abelian.

We start our analysis by introducing a new basis for twining characters. Namely, we define the following linear combination

$$\chi^{ab}_{(\lambda;\rho)}(q) = \sqrt{\frac{\dim\rho}{|\mathcal{S}(\lambda)|}} \sum_{\omega\in\mathcal{S}(\lambda)} \overline{\rho^{ab}(\omega)}\, \chi^{\omega}_{\lambda}(q) = \sqrt{\frac{\dim\rho}{|\mathcal{S}(\lambda)|}}\, \mathrm{Tr}_\lambda\Big(\sum_{\omega\in\mathcal{S}(\lambda)} R(\overline{\rho^{ab}(\omega)}\,\omega)\, q^{L_0-\frac{c}{24}} \Big)\,. \tag{3.4}$$

Here $\mathcal{S}(\lambda) \subset G$ is the stabilizer of λ defined in (2.35), and $\rho \in \mathrm{Irr}(\mathcal{S}(\lambda))$ is a unitary irreducible representation of $\mathcal{S}(\lambda)$,

$$\rho(\omega) = \big(\rho^{ab}(\omega)\big)_{1\le a,\,b\le n} \in U(n;\mathbb{C}) \quad (n = \dim\rho)\,. \tag{3.5}$$

The set $\{\chi^{ab}_{(\lambda;\rho)}\,|\,\rho\in\mathrm{Irr}(\mathcal{S}(\lambda));\,1\le a,b\le\dim\rho\}$ forms a basis for the twining characters of $\lambda\in\mathcal{I}$. Actually, one can express χ^ω_λ in terms of $\chi^{ab}_{(\lambda;\rho)}$,

$$\chi^\omega_\lambda(q)=\frac{1}{\sqrt{|\mathcal{S}(\lambda)|}}\sum_{\substack{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))\\ 1\le a,b\le\dim\rho}}\sqrt{\dim\rho}\,\rho^{ab}(\omega)\,\chi^{ab}_{(\lambda;\rho)}(q)\,. \tag{3.6}$$

Here we used the orthogonality relations for matrix elements of irreducible representations

$$\frac{1}{|H|}\sum_{h\in H}\rho_i^{ab}(h)\,\overline{\rho_j^{cd}(h)}=\frac{\delta_{ij}\delta^{ac}\delta^{bd}}{\dim\rho_i}\quad(\rho_i,\rho_j\in\mathrm{Irr}(H)), \tag{3.7a}$$

$$\frac{1}{|H|}\sum_{\substack{\rho\in\mathrm{Irr}(H)\\ 1\le a,b\le\dim\rho}}(\dim\rho)\,\rho^{ab}(h)\,\overline{\rho^{ab}(h')}=\delta_{hh'}\quad(h,h'\in H)\,, \tag{3.7b}$$

which hold for any finite group H.

In terms of $\chi^{ab}_{(\lambda;\rho)}$, the modular transformation of the character $\chi_L\,(L\in\hat{\mathcal{I}})$ can be expressed as follows,

$$\begin{aligned}\chi_L(q)&=\sum_{\lambda\in\mathcal{I}(\omega_L)}S^{\omega_L}_{L\lambda}\chi^{\omega_L}_\lambda(\tilde q)\\ &=\sum_{\lambda\in\mathcal{I}(\omega_L)}S^{\omega_L}_{L\lambda}\frac{1}{\sqrt{|\mathcal{S}(\lambda)|}}\sum_{\substack{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))\\ 1\le a,b\le\dim\rho}}\sqrt{\dim\rho}\,\rho^{ab}(\omega_L)\,\chi^{ab}_{(\lambda;\rho)}(\tilde q)\\ &=\sum_{\lambda\in\mathcal{I}(\omega_L)}\sum_{\substack{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))\\ 1\le a,b\le\dim\rho}}S^{\omega_L}_{L\lambda}\sqrt{\frac{\dim\rho}{|\mathcal{S}(\lambda)|}}\rho^{ab}(\omega_L)\,\chi^{ab}_{(\lambda;\rho)}(\tilde q)\,.\end{aligned} \tag{3.8}$$

We write this in the following form,

$$\chi_L(q)=\sum_{(\lambda;\rho)\in\hat{\mathcal{I}}^*}\sum_{1\le a,b\le\dim\rho}\hat S^{ab}_{L(\lambda;\rho)}\chi^{ab}_{(\lambda;\rho)}(\tilde q)\,. \tag{3.9}$$

Here the set $\hat{\mathcal{I}}^*$ and the matrix $\hat S$ are defined as follows,

$$\hat{\mathcal{I}}^*=\{(\lambda;\rho)\,|\,\lambda\in\mathcal{I};\,\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))\}\,, \tag{3.10}$$

$$\hat S^{ab}_{L(\lambda;\rho)}=\begin{cases}S^{\omega_L}_{L\lambda}\sqrt{\frac{\dim\rho}{|\mathcal{S}(\lambda)|}}\rho^{ab}(\omega_L) & (\lambda\in\mathcal{I}(\omega_L),\ \rho\in\mathrm{Irr}(\mathcal{S}(\lambda)))\,,\\ 0 & (\lambda\notin\mathcal{I}(\omega_L)).\end{cases} \tag{3.11}$$

We call $\hat S$ the generalized S-matrix of the chiral algebra $\mathcal{A}$. This matrix $\hat S$ is square since

$$|\hat{\mathcal{I}}|=\sum_{\omega\in G}|\mathcal{I}^\omega|=\sum_{\omega\in G}|\mathcal{I}(\omega)|=\sum_{\lambda\in\mathcal{I}}|\mathcal{S}(\lambda)|=\sum_{\lambda\in\mathcal{I}}\sum_{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))}(\dim\rho)^2=\sum_{(\lambda;\rho)\in\hat{\mathcal{I}}^*}(\dim\rho)^2\,. \tag{3.12}$$

Moreover, $\hat{S}$ is unitary,

$$\begin{aligned}
\sum_{(\lambda;\rho)\in\hat{\mathcal{I}}^*;a,b} \hat{S}^{ab}_{L(\lambda;\rho)}\,\overline{\hat{S}^{ab}_{M(\lambda;\rho)}} &= \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)} S^{\omega_L}_{L\lambda}\,\overline{S^{\omega_M}_{M\lambda}} \sum_{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))}\sum_{a,b}\frac{\dim\rho}{|\mathcal{S}(\lambda)|}\rho^{ab}(\omega_L)\overline{\rho^{ab}(\omega_M)} \\
&= \sum_{\lambda\in\mathcal{I}(\omega_L)} S^{\omega_L}_{L\lambda}\,\overline{S^{\omega_L}_{M\lambda}}\times\delta_{\omega_L\omega_M} \\
&= \delta_{LM}\,,
\end{aligned} \tag{3.13}$$

where we used the orthogonality relation (3.7b). The same matrix for the case of $G\cong\mathbb{Z}_2$ is found in [9, 10].

One can express the generalized fusion coefficients (2.62) in a form completely parallel to the ordinary Verlinde formula (3.1) using the generalized S-matrix (3.11) in place of the ordinary S-matrix. Namely, we can prove the following formula

$$\widehat{\mathcal{N}}_{LM}{}^N = \sum_{(\lambda;\rho)\in\hat{\mathcal{I}}^*;a,b,c}\frac{\hat{S}^{ab}_{L(\lambda;\rho)}\hat{S}^{bc}_{M(\lambda;\rho)}\overline{\hat{S}^{ac}_{N(\lambda;\rho)}}}{\hat{S}^{11}_{0(\lambda;\rho)}}\,. \tag{3.14}$$

This can be readily checked by a straightforward calculation,

$$\begin{aligned}
&\text{r.h.s. of (3.14)} \\
&= \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)\cap\mathcal{I}(\omega_N)}\frac{S^{\omega_L}_{L\lambda}\,S^{\omega_M}_{M\lambda}\,\overline{S^{\omega_N}_{N\lambda}}}{S_{0\lambda}}\sum_{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))}\sum_{a,b,c}\frac{\dim\rho}{|\mathcal{S}(\lambda)|}\rho^{ab}(\omega_L)\rho^{bc}(\omega_M)\overline{\rho^{ac}(\omega_N)} \\
&= \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)\cap\mathcal{I}(\omega_N)}\frac{S^{\omega_L}_{L\lambda}\,S^{\omega_M}_{M\lambda}\,\overline{S^{\omega_N}_{N\lambda}}}{S_{0\lambda}}\sum_{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))}\sum_{a}\frac{\dim\rho}{|\mathcal{S}(\lambda)|}\rho^{aa}(\omega_L\omega_M\omega_N^{-1}) \\
&= \sum_{\lambda\in\mathcal{I}(\omega_L)\cap\mathcal{I}(\omega_M)}\frac{S^{\omega_L}_{L\lambda}\,S^{\omega_M}_{M\lambda}\,\overline{S^{\omega_N}_{N\lambda}}}{S_{0\lambda}}\times\delta_{\omega_L\omega_M,\omega_N} \\
&= \widehat{\mathcal{N}}_{LM}{}^N\,,
\end{aligned} \tag{3.15}$$

where we used the orthogonality relation (3.7b) for $h'=1$. We call eq.(3.14) the generalized Verlinde formula.

The similarity of eq.(3.14) to the ordinary Verlinde formula (3.1) suggests that the quantum dimension (3.2) is also generalized to the case of generalized fusion algebras. Actually, one can prove the following

$$\mathfrak{b}_{(\lambda;\rho)}(L)\,\mathfrak{b}_{(\lambda;\rho)}(M) = \sum_{N\in\hat{\mathcal{I}}}\widehat{\mathcal{N}}_{LM}{}^N\,\mathfrak{b}_{(\lambda;\rho)}(N)\,, \tag{3.16}$$

where $\mathfrak{b}_{(\lambda;\rho)}(L)$ is a matrix defined as

$$\mathfrak{b}_{(\lambda;\rho)}(L) = \left(\frac{\hat{S}^{ab}_{L(\lambda;\rho)}}{\hat{S}^{11}_{0(\lambda;\rho)}}\right)_{1\le a,b\le n}\in M_n(\mathbb{C})\quad (n=\dim\rho,(\lambda;\rho)\in\hat{\mathcal{I}}^*)\,. \tag{3.17}$$

The proof of eq.(3.16) is straightforward,

$$\begin{aligned}
\sum_{N\in\hat{\mathcal{I}}} \widehat{\mathcal{N}}_{LM}{}^{N}\,\mathfrak{b}^{ab}_{(\lambda;\rho)}(N) &= \sum_{N\in\hat{\mathcal{I}}}\ \sum_{(\lambda';\rho')\in\hat{\mathcal{I}}^*;a',b',c'} \frac{\hat{S}^{a'b'}_{L(\lambda';\rho')}\hat{S}^{b'c'}_{M(\lambda';\rho')}\overline{\hat{S}^{a'c'}_{N(\lambda';\rho')}}}{\hat{S}^{11}_{0(\lambda';\rho')}}\ \frac{\hat{S}^{ab}_{N(\lambda;\rho)}}{\hat{S}^{11}_{0(\lambda;\rho)}} \\
&= \sum_{(\lambda';\rho')\in\hat{\mathcal{I}}^*;a',b',c'} \frac{\hat{S}^{a'b'}_{L(\lambda';\rho')}\hat{S}^{b'c'}_{M(\lambda';\rho')}}{\hat{S}^{11}_{0(\lambda';\rho')}\hat{S}^{11}_{0(\lambda;\rho)}}\,\delta_{\lambda'\lambda}\delta_{\rho'\rho}\delta^{a'a}\delta^{c'b} \\
&= \sum_{b'} \frac{\hat{S}^{ab'}_{L(\lambda;\rho)}}{\hat{S}^{11}_{0(\lambda;\rho)}}\ \frac{\hat{S}^{b'b}_{M(\lambda;\rho)}}{\hat{S}^{11}_{0(\lambda;\rho)}} \\
&= \sum_{b'} \mathfrak{b}^{ab'}_{(\lambda;\rho)}(L)\,\mathfrak{b}^{b'b}_{(\lambda;\rho)}(M)\,.
\end{aligned} \tag{3.18}$$

The above equation (3.16) means that $\mathfrak{b}_{(\lambda;\rho)}$ defined in (3.17) realizes a representation of the generalized fusion algebra $\mathcal{F}(\mathcal{A};G)$ by linearly extending its action on the basis (L) to the entire algebra. For the case of ordinary fusion algebras, $\mathfrak{b}_{(\lambda;\rho)}$ reduces to the quantum dimension b_λ since the generalized S-matrix $\hat{S}$ coincides with the ordinary S-matrix for ordinary fusion algebras. Therefore $\mathfrak{b}_{(\lambda;\rho)}$ actually gives a generalization of the one-dimensional representations b_λ to the case of generalized fusion algebras.

Since we define the generalized fusion algebra by products of the boundary vertex operators, we can expect that the generalized fusion algebra is associative. However the associativity is not manifest from the definition (2.62) of the generalized fusion coefficients and its proof is desirable. Let $\widehat{N}_L$ be a matrix $(\widehat{N}_L)_M{}^N = \widehat{\mathcal{N}}_{ML}{}^N$. The associativity of the generalized fusion algebra is equivalent with the following condition for $\widehat{N}_L$,

$$\widehat{N}_L\widehat{N}_M = \sum_{N\in\hat{\mathcal{I}}} \widehat{\mathcal{N}}_{LM}{}^{N}\widehat{N}_N\,. \tag{3.19}$$

One can easily see that this follows from the generalized Verlinde formula (3.14). First note that

$$\begin{aligned}
(\widehat{N}_L)_M{}^N = \widehat{\mathcal{N}}_{ML}{}^N &= \sum_{(\lambda;\rho);a,b}\ \sum_{(\lambda';\rho');a',b'} \hat{S}^{ab}_{M(\lambda;\rho)}\,\delta_{\lambda\lambda'}\delta_{\rho\rho'}\delta^{aa'}\frac{\hat{S}^{bb'}_{L(\lambda;\rho)}}{\hat{S}^{11}_{0(\lambda;\rho)}}\,\overline{\hat{S}^{a'b'}_{N(\lambda';\rho')}} \\
&= \sum_{(\lambda;\rho);a,b}\ \sum_{(\lambda';\rho');a',b'} \hat{S}^{ab}_{M(\lambda;\rho)}\,\delta_{\lambda\lambda'}\delta_{\rho\rho'}\delta^{aa'}\,\mathfrak{b}^{bb'}_{(\lambda;\rho)}(L)\,\overline{\hat{S}^{a'b'}_{N(\lambda';\rho')}} \\
&= \left[\hat{S}\ \bigoplus_{(\lambda;\rho)}\Big(\underbrace{\mathfrak{b}_{(\lambda;\rho)}(L)\oplus\cdots\oplus\mathfrak{b}_{(\lambda;\rho)}(L)}_{(\dim\rho)\ \text{terms}}\Big)\ \hat{S}^\dagger\right]_M^{\ N},
\end{aligned} \tag{3.20}$$

which shows that the matrix $\widehat{N}_L$ is similar to a direct sum of $\mathfrak{b}_{(\lambda;\rho)}(L)$. The equation (3.19) is an immediate consequence of this fact since $\mathfrak{b}_{(\lambda;\rho)}(L)$ satisfies the generalized fusion algebra (3.16) for all $(\lambda;\rho)\in\hat{\mathcal{I}}^*$. The generalized fusion algebra is therefore associative and the matrix $\widehat{N}_L$ realizes the (right) regular representation of the generalized fusion algebra.

We next show the mutual independence, in particular irreducibility, of $\mathfrak{b}_{(\lambda;\rho)}$ for $(\lambda;\rho) \in \hat{\mathcal{I}}^*$. To see this, we use the explicit form of $\mathfrak{b}_{(\lambda;\rho)}(L)$,

$$\mathfrak{b}_{(\lambda;\rho)}(L) = \begin{cases} \frac{S^{\omega_L}_{L\lambda}}{S_{0\lambda}} \rho(\omega_L) & (\omega_L \in \mathcal{S}(\lambda))\,, \\ 0 & (\text{otherwise})\,, \end{cases} \tag{3.21}$$

which follows from eq.(3.11). Suppose that $\mathfrak{b}_{(\lambda;\rho)}$ is reducible. Then there exists a matrix X such that

$$X \mathfrak{b}_{(\lambda;\rho)}(L) X^{-1} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \text{for all } L \in \hat{\mathcal{I}}\,. \tag{3.22}$$

However this implies that $\rho \in \mathrm{Irr}(\mathcal{S}(\lambda))$ is also reducible, since there exists at least one representation $L \in \mathcal{I}^\omega$ for each $\omega \in \mathcal{S}(\lambda)$ such that $S^\omega_{L\lambda} \neq 0$ and $\rho(\omega) = \frac{S_{0\lambda}}{S^\omega_{L\lambda}} \mathfrak{b}_{(\lambda;\rho)}(L)$. This contradiction proves that $\mathfrak{b}_{(\lambda;\rho)}$ is irreducible for all $(\lambda;\rho) \in \hat{\mathcal{I}}^*$.

In order to show the mutual independence of $\mathfrak{b}_{(\lambda;\rho)}$, we introduce the character of $\mathfrak{b}_{(\lambda;\rho)}$,

$$\hat{b}_{(\lambda;\rho)}(L) = \sum_a \mathfrak{b}^{aa}_{(\lambda;\rho)}(L) = \begin{cases} \frac{S^{\omega_L}_{L\lambda}}{S_{0\lambda}} \psi(\omega_L) & (\omega_L \in \mathcal{S}(\lambda))\,, \\ 0 & (\text{otherwise})\,, \end{cases} \tag{3.23}$$

where ψ is the group character of the representation $\rho \in \mathrm{Irr}(\mathcal{S}(\lambda))$. These characters satisfy the following orthogonality relation

$$\sum_{L \in \hat{\mathcal{I}}} \hat{b}_{(\lambda;\rho)}(L)\, \overline{\hat{b}_{(\lambda';\rho')}(L)} = \sum_{a,b} \sum_L \frac{\hat{S}^{aa}_{L(\lambda;\rho)}}{\hat{S}^{11}_{0(\lambda;\rho)}} \frac{\overline{\hat{S}^{bb}_{L(\lambda';\rho')}}}{\hat{S}^{11}_{0(\lambda';\rho')}} = \sum_a \frac{\delta_{\lambda\lambda'}\delta_{\rho\rho'}\delta^{aa}}{(S_{0\lambda})^2 \frac{\dim\rho}{|\mathcal{S}(\lambda)|}} = \frac{\delta_{\lambda\lambda'}\delta_{\rho\rho'}}{\frac{1}{|\mathcal{S}(\lambda)|}(S_{0\lambda})^2}\,. \tag{3.24}$$

Suppose that two representations $\mathfrak{b}_{(\lambda;\rho)}$ and $\mathfrak{b}_{(\lambda';\rho')}$ are similar, *i.e.*, there exists a matrix Y such that

$$\mathfrak{b}_{(\lambda;\rho)}(L) = Y \mathfrak{b}_{(\lambda';\rho')}(L) Y^{-1} \quad \text{for all } L \in \hat{\mathcal{I}}\,. \tag{3.25}$$

Then the corresponding characters are the same, $\hat{b}_{(\lambda;\rho)}(L) = \hat{b}_{(\lambda';\rho')}(L)$. However this implies that $(\lambda;\rho) = (\lambda';\rho')$ since

$$\sum_{L \in \hat{\mathcal{I}}} \hat{b}_{(\lambda;\rho)}(L)\, \overline{\hat{b}_{(\lambda';\rho')}(L)} = \sum_L |\hat{b}_{(\lambda;\rho)}(L)|^2 > 0 \tag{3.26}$$

which is not possible for $(\lambda;\rho) \neq (\lambda';\rho')$ from the orthogonality relation (3.24). Therefore two representations $\mathfrak{b}_{(\lambda;\rho)}$ and $\mathfrak{b}_{(\lambda';\rho')}$ are distinct if $(\lambda;\rho) \neq (\lambda';\rho')$. To summarize, we have shown that $\{\mathfrak{b}_{(\lambda;\rho)} | (\lambda;\rho) \in \hat{\mathcal{I}}^*\}$ is a set of mutually independent irreducible representations of the generalized fusion algebra.

We illustrate our results by the generalized fusion algebra $\mathcal{F}(so(8)_1; S_3)$ presented in Section 2.3. The stabilizer (2.35) for the representations (2.63) of $so(8)_1$ is determined from the action (2.65) of S_3 on the representations,

$$\mathcal{S}(O) = S_3\,, \quad \mathcal{S}(V) = \{1, \sigma\}\,, \quad \mathcal{S}(S) = \{1, \pi^2\sigma\}\,, \quad \mathcal{S}(C) = \{1, \pi\sigma\}\,. \tag{3.27}$$

	1	π	π^2	σ	$\pi\sigma$	$\pi^2\sigma$
ρ_0	1	1	1	1	1	1
ρ_1	1	1	1	-1	-1	-1
ρ_2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix}$	$\begin{pmatrix} \bar{\kappa} & 0 \\ 0 & \kappa \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \kappa \\ \bar{\kappa} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \bar{\kappa} \\ \kappa & 0 \end{pmatrix}$

Table 2: Irreducible representations ρ_0, ρ_1, ρ_2 of the symmetric group $S_3 = \langle \pi, \sigma \rangle$. κ is a cube root of 1, $\kappa = e^{\frac{2\pi i}{3}}$.

The stabilizers for V, S, C are isomorphic to $\mathbb{Z}_2$, for which there are two irreducible representations. We denote them as $\rho_\pm \in \mathrm{Irr}(\mathbb{Z}_2)$, which are defined for the case of $\mathcal{S}(V)$ as

$$\rho_\pm(1) = 1 \,, \quad \rho_\pm(\sigma) = \pm 1 \,. \tag{3.28}$$

For the symmetric group S_3, there are three irreducible representations, ρ_0, ρ_1 and ρ_2, two of which are one-dimensional while the remaining one is two-dimensional (see Table 2). Accordingly, the set $\hat{\mathcal{I}}^*$ in (3.10) takes the form

$$\begin{aligned} \hat{\mathcal{I}}^* &= \{(\lambda; \rho) \,|\, \lambda \in \mathcal{I};\, \rho \in \mathrm{Irr}(\mathcal{S}(\lambda))\} \\ &= \{(O;0), (O;1), (O;2), (V;+), (V;-), (S;+), (S;-), (C;+), (C;-)\} \,, \end{aligned} \tag{3.29}$$

where we write ρ_a ($a = 0, 1, 2; \pm$) as a for simplicity. The generalized S-matrix (3.11), which is a 12×12 matrix in the present case, is then obtained in the following form,

$$\hat{S} = \frac{1}{2\sqrt{6}} \times \begin{array}{c|cccccccccccc} & (O;0) & (O;1) & (O;2)_{11} & (O;2)_{22} & (O;2)_{12} & (O;2)_{21} & (V;+) & (V;-) & (S;+) & (S;-) & (C;+) & (C;-) \\ \hline O & 1 & 1 & \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} \\ V & 1 & 1 & \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{3} & \sqrt{3} & -\sqrt{3} & -\sqrt{3} & -\sqrt{3} & -\sqrt{3} \\ S & 1 & 1 & \sqrt{2} & \sqrt{2} & 0 & 0 & -\sqrt{3} & -\sqrt{3} & \sqrt{3} & \sqrt{3} & -\sqrt{3} & -\sqrt{3} \\ C & 1 & 1 & \sqrt{2} & \sqrt{2} & 0 & 0 & -\sqrt{3} & -\sqrt{3} & -\sqrt{3} & -\sqrt{3} & \sqrt{3} & \sqrt{3} \\ (0)_\pi & 2 & 2 & \sqrt{8}\kappa & \sqrt{8}\bar{\kappa} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (0)_{\pi^2} & 2 & 2 & \sqrt{8}\bar{\kappa} & \sqrt{8}\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (0)_\sigma & \sqrt{2} & -\sqrt{2} & 0 & 0 & 2 & 2 & \sqrt{6} & -\sqrt{6} & 0 & 0 & 0 & 0 \\ (1)_\sigma & \sqrt{2} & -\sqrt{2} & 0 & 0 & 2 & 2 & -\sqrt{6} & \sqrt{6} & 0 & 0 & 0 & 0 \\ (0)_{\pi\sigma} & \sqrt{2} & -\sqrt{2} & 0 & 0 & 2\kappa & 2\bar{\kappa} & 0 & 0 & 0 & 0 & \sqrt{6} & -\sqrt{6} \\ (1)_{\pi\sigma} & \sqrt{2} & -\sqrt{2} & 0 & 0 & 2\kappa & 2\bar{\kappa} & 0 & 0 & 0 & 0 & -\sqrt{6} & \sqrt{6} \\ (0)_{\pi^2\sigma} & \sqrt{2} & -\sqrt{2} & 0 & 0 & 2\bar{\kappa} & 2\kappa & 0 & 0 & \sqrt{6} & -\sqrt{6} & 0 & 0 \\ (1)_{\pi^2\sigma} & \sqrt{2} & -\sqrt{2} & 0 & 0 & 2\bar{\kappa} & 2\kappa & 0 & 0 & -\sqrt{6} & \sqrt{6} & 0 & 0 \end{array} \,, \tag{3.30}$$

where $\kappa = e^{\frac{2\pi i}{3}}$. One can readily check that this matrix is unitary and that the generalized Verlinde formula (3.14) reproduces the generalized fusion coefficients of $\mathcal{F}(so(8)_1; S_3)$ given in Table 1. From this matrix $\hat{S}$, one can construct the irreducible representations (3.17) of $\mathcal{F}(so(8)_1; S_3)$. We give the result in Table 3. From this table, one can check that $\mathfrak{b}_{(\lambda;\rho)}(L)\,((\lambda;\rho) \in \hat{\mathcal{I}}^*)$ indeed satisfies the generalized fusion algebra in Table 1 and realizes a representation of $\mathcal{F}(so(8)_1; S_3)$.

Before concluding this section, we comment on the case that the automorphism group G is abelian. Since all the irreducible representations are one-dimensional for an abelian G, the representation matrix $\rho(\omega)$ has only one component $\rho^{11}(\omega)$. Therefore we can omit the suffix and write $\rho(\omega)$ instead of $\rho^{11}(\omega)$. This enables us to express the formulas we have given above in a simple form. First, the basis (3.4) of twining characters and its modular transformation can be written as follows,

$$\chi_{(\lambda;\rho)}(q) = \frac{1}{\sqrt{|\mathcal{S}(\lambda)|}} \sum_{\omega \in \mathcal{S}(\lambda)} \overline{\rho(\omega)}\, \chi_\lambda^\omega(q)\,, \tag{3.31}$$

$$\chi_L(q) = \sum_{(\lambda;\rho)\in\hat{\mathcal{I}}^*} \hat{S}_{L(\lambda;\rho)} \chi_{(\lambda;\rho)}(\tilde{q})\,, \tag{3.32}$$

where the generalized S-matrix is defined as

$$\hat{S}_{L(\lambda;\rho)} = \begin{cases} S_{L\lambda}^{\omega_L} \frac{1}{\sqrt{|\mathcal{S}(\lambda)|}} \rho(\omega_L) & (\lambda \in \mathcal{I}(\omega_L),\ \rho \in \mathrm{Irr}(\mathcal{S}(\lambda)))\,, \\ 0 & (\lambda \notin I(\omega_L)). \end{cases} \tag{3.33}$$

The generalized Verlinde formula (3.14) then takes the form

$$\widehat{\mathcal{N}}_{LM}{}^{N} = \sum_{(\lambda;\rho)\in\hat{\mathcal{I}}^*} \frac{\hat{S}_{L(\lambda;\rho)}\hat{S}_{M(\lambda;\rho)}\overline{\hat{S}_{N(\lambda;\rho)}}}{\hat{S}_{0(\lambda;\rho)}}\,. \tag{3.34}$$

For $G \cong \mathbb{Z}_2$, the generalized S-matrix (3.33) and the generalized Verlinde formula (3.34) reproduce the result given in [9, 10]. The matrix $\hat{S}$ is denoted as $\tilde{S}$ in [9], which is considered to be a particular case of the same matrix in [8] for a general finite abelian G. This suggests that two matrices $\hat{S}$ in (3.33) and $\tilde{S}$ in [8] are the same, although we have no proof for this statement. If this is the case, the classifying algebra in [8] is the dual of the generalized fusion algebra (3.34) in the sense of C-algebras [18, 19, 20] with the structure constants

$$M_{(\lambda;\rho)(\lambda';\rho')}{}^{(\lambda'';\rho'')} = \sum_{L\in\hat{\mathcal{I}}} \frac{\hat{S}_{L(\lambda;\rho)}\hat{S}_{L(\lambda';\rho')}\overline{\hat{S}_{L(\lambda'';\rho'')}}}{\hat{S}_{L(0;\rho_0)}}\,, \tag{3.35}$$

where ρ_0 is the identity representation $\rho_0(\omega) = 1\,(\forall\omega \in G)$ of G. Clearly, the irreducible representation of this algebra takes the form $(\lambda;\rho) \mapsto \frac{\hat{S}_{L(\lambda;\rho)}}{\hat{S}_{L(0;\rho_0)}}$ and is labeled by $L \in \hat{\mathcal{I}}$.

	O	V	S	C	$(0)_\pi$	$(0)_{\pi^2}$	$(0)_\sigma$	$(1)_\sigma$	$(0)_{\pi\sigma}$	$(1)_{\pi\sigma}$	$(0)_{\pi^2\sigma}$	$(1)_{\pi^2\sigma}$
$\mathfrak{b}_{(O;0)}$	1	1	1	1	2	2	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
$\mathfrak{b}_{(O;1)}$	1	1	1	1	2	2	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$
$\mathfrak{b}_{(O;2)}$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$2\left(\begin{smallmatrix}\kappa&0\\0&\bar\kappa\end{smallmatrix}\right)$	$2\left(\begin{smallmatrix}\bar\kappa&0\\0&\kappa\end{smallmatrix}\right)$	$\sqrt{2}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\sqrt{2}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$	$\sqrt{2}\left(\begin{smallmatrix}0&\kappa\\\bar\kappa&0\end{smallmatrix}\right)$	$\sqrt{2}\left(\begin{smallmatrix}0&\kappa\\\bar\kappa&0\end{smallmatrix}\right)$	$\sqrt{2}\left(\begin{smallmatrix}0&\bar\kappa\\\kappa&0\end{smallmatrix}\right)$	$\sqrt{2}\left(\begin{smallmatrix}0&\bar\kappa\\\kappa&0\end{smallmatrix}\right)$
$\mathfrak{b}_{(V;+)}$	1	1	-1	-1	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	0
$\mathfrak{b}_{(V;-)}$	1	1	-1	-1	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	0
$\mathfrak{b}_{(S;+)}$	1	-1	1	-1	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$
$\mathfrak{b}_{(S;-)}$	1	-1	1	-1	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$
$\mathfrak{b}_{(C;+)}$	1	-1	-1	1	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	0
$\mathfrak{b}_{(C;-)}$	1	-1	-1	1	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	0

Table 3: Irreducible representations of the generalized fusion algebra $\mathcal{F}(so(8)_1; S_3)$. κ is a cube root of 1, $\kappa = e^{\frac{2\pi i}{3}}$. Unlike ordinary fusion algebras, there is an irreducible representation of dimension 2 due to the non-commutativity of $\mathcal{F}(so(8)_1; S_3)$.

4 Boundary states as a NIM-rep of generalized fusion algebras

It is now well understood that boundary states preserving a chiral algebra $\mathcal{A}$ form a non-negative integer matrix representation (NIM-rep) of the fusion algebra $\mathcal{F}(\mathcal{A})$ [2]. Since the generalized fusion algebra $\mathcal{F}(\mathcal{A}; G)$ is defined through the twisted boundary states associated with the charge-conjugation modular invariant, it is natural to expect a close relationship between $\mathcal{F}(\mathcal{A}; G)$ and twisted boundary states for general modular invariants, such as simple current invariants [16] or exceptional ones. As we shall show below, this expectation turns out to be true; a set of mutually consistent boundary states associated with any modular invariant realizes a NIM-rep of the generalized fusion algebra.

4.1 Charge-conjugation invariants

We begin our analysis by rewriting the twisted boundary states for the charge-conjugation modular invariant in the form that the relation to the generalized fusion algebra is more apparent. We call them the regular states since their mutual overlap is expressed by the generalized fusion coefficients, which realize the regular representation of the generalized fusion algebra.

Corresponding to the character $\chi^{ab}_{(\lambda;\rho)}$ defined in eq.(3.4), we introduce a new basis for the twisted Ishibashi states,

$$|((\lambda, \lambda^*); \rho)_{ab}\rangle\rangle = \sqrt{\frac{\dim \rho}{|\mathcal{S}(\lambda)|}} \sum_{\omega \in \mathcal{S}(\lambda)} \overline{\rho^{ab}(\omega)}\, |(\lambda, \lambda^*); \omega\rangle\rangle \quad (\rho \in \mathrm{Irr}(\mathcal{S}(\lambda)))\,. \tag{4.1}$$

One can express the original basis in terms of $|((\lambda,\lambda^*);\rho)_{ab}\rangle\!\rangle$ as

$$|(\lambda,\lambda^*);\omega\rangle\!\rangle = \frac{1}{\sqrt{|\mathcal{S}(\lambda)|}} \sum_{\substack{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))\\ 1\le a,b\le \dim\rho}} \sqrt{\dim\rho}\,\rho^{ab}(\omega)\,|((\lambda,\lambda^*);\rho)_{ab}\rangle\!\rangle\,, \tag{4.2}$$

where we used the orthogonality relation (3.7b) for the matrix elements of ρ. We denote the set of labels for the new basis $|((\lambda,\lambda^*);\rho)_{ab}\rangle\!\rangle$ of twisted Ishibashi states by $\hat{\mathcal{E}}_0$,

$$\hat{\mathcal{E}}_0 = \{((\lambda,\lambda^*);\rho)\,|\,\lambda\in\mathcal{I};\ \rho\in\mathrm{Irr}(\mathcal{S}(\lambda))\}\,. \tag{4.3}$$

This is essentially the same set as $\hat{\mathcal{I}}^*$ defined in (3.10). We have introduced the new symbol $\hat{\mathcal{E}}_0$ for the labels of Ishibashi states in order to reserve $\hat{\mathcal{I}}^*$ for expressing the chiral quantity. The overlap of $|((\lambda,\lambda^*);\rho)_{ab}\rangle\!\rangle$ with $|((\lambda,\lambda^*);\rho')_{a'b'}\rangle\!\rangle$ yields $\chi^{b'b}_{(\lambda;\rho)}$,

$$\begin{aligned}
\langle\!\langle((\lambda,\lambda^*);\rho')_{a'b'}|\tilde{q}^{\frac{1}{2}H_c}|((\lambda,\lambda^*);\rho)_{ab}\rangle\!\rangle &= \frac{\sqrt{\dim\rho'}\sqrt{\dim\rho}}{|\mathcal{S}(\lambda)|} \sum_{\omega,\omega'\in\mathcal{S}(\lambda)} \rho'^{a'b'}(\omega')\overline{\rho^{ab}(\omega)}\frac{1}{S_{0\lambda}}\chi_\lambda^{\omega'^{-1}\omega}(\tilde{q})\\
&= \frac{\sqrt{\dim\rho'}\sqrt{\dim\rho}}{|\mathcal{S}(\lambda)|} \sum_{\omega,\omega'\in\mathcal{S}(\lambda)} \rho'^{a'b'}(\omega')\overline{\rho^{ab}(\omega'\omega)}\frac{1}{S_{0\lambda}}\chi_\lambda^{\omega}(\tilde{q})\\
&= \delta_{\rho\rho'}\delta^{aa'}\sum_{\omega\in\mathcal{S}(\lambda)}\overline{\rho^{b'b}(\omega)}\frac{1}{S_{0\lambda}}\chi_\lambda^{\omega}(\tilde{q})\\
&= \delta_{\rho\rho'}\delta^{aa'}\frac{1}{S_{0\lambda}}\sqrt{\frac{|\mathcal{S}(\lambda)|}{\dim\rho}}\,\chi^{b'b}_{(\lambda;\rho)}(\tilde{q})\\
&= \delta_{\rho\rho'}\delta^{aa'}\frac{1}{\hat{S}^{11}_{0(\lambda;\rho)}}\chi^{b'b}_{(\lambda;\rho)}(\tilde{q})\,.
\end{aligned} \tag{4.4}$$

Using $|((\lambda,\lambda^*);\rho)_{ab}\rangle\!\rangle$, we can rewrite the regular state $|L\rangle$ ($L\in\hat{\mathcal{I}}$) as follows

$$\begin{aligned}
|L\rangle &= \sum_{\lambda\in\mathcal{I}(\omega_L)} S^{\omega_L}_{L\lambda}|(\lambda,\lambda^*);\omega_L\rangle\!\rangle\\
&= \sum_{\lambda\in\mathcal{I}(\omega_L)} S^{\omega_L}_{L\lambda}\frac{1}{\sqrt{|\mathcal{S}(\lambda)|}}\sum_{\substack{\rho\in\mathrm{Irr}(\mathcal{S}(\lambda))\\ 1\le a,b\le\dim\rho}}\sqrt{\dim\rho}\,\rho^{ab}(\omega_L)|((\lambda,\lambda^*);\rho)_{ab}\rangle\!\rangle\\
&= \sum_{((\lambda,\lambda^*);\rho)\in\hat{\mathcal{E}}_0}\sum_{a,b}\hat{S}^{ab}_{L(\lambda;\rho)}|((\lambda,\lambda^*);\rho)_{ab}\rangle\!\rangle\,.
\end{aligned} \tag{4.5}$$

The boundary state coefficients of the regular state $|L\rangle$ are therefore given by the generalized S-matrix $\hat{S}$. It is instructive to calculate the overlaps of $|L\rangle$ starting from the form given

above,

$$
\begin{aligned}
\langle M|\tilde{q}^{\frac{1}{2}H_c}|L\rangle &= \sum_{(\lambda;\rho)\in\hat{\mathcal{I}}^*}\sum_{a,b}\sum_{a',b'}\overline{\hat{S}^{a'b'}_{M(\lambda;\rho)}}\hat{S}^{ab}_{L(\lambda;\rho)}\delta^{aa'}\frac{1}{\hat{S}^{11}_{0(\lambda;\rho)}}\chi^{b'b}_{(\lambda;\rho)}(\tilde{q}) \\
&= \sum_{(\lambda;\rho)}\sum_{a,b}\sum_{b'}\overline{\hat{S}^{ab'}_{M(\lambda;\rho)}}\hat{S}^{ab}_{L(\lambda;\rho)}\frac{1}{\hat{S}^{11}_{0(\lambda;\rho)}}\sum_{N}\overline{\hat{S}^{b'b}_{N(\lambda;\rho)}}\,\chi_N(q) \\
&= \sum_{N}\sum_{(\lambda;\rho);a,b,b'}\frac{\hat{S}^{ab}_{L(\lambda;\rho)}\hat{S}^{bb'}_{N^*(\lambda;\rho)}\overline{\hat{S}^{ab'}_{M(\lambda;\rho)}}}{\hat{S}^{11}_{0(\lambda;\rho)}}\,\chi_N(q) \\
&= \sum_{N}\widehat{\mathcal{N}}_{LN}{}^{M}\chi_{N^*}(q) \\
&= \sum_{N}(\widehat{N}_N)_L{}^{M}\chi_{N^*}(q)\,,
\end{aligned}
\tag{4.6}
$$

where we used the formula

$$
\overline{\hat{S}^{ab}_{L(\lambda;\rho)}} = \hat{S}^{ba}_{L^*(\lambda;\rho)}
\tag{4.7}
$$

which follows from eq.(2.59). In this way, we reproduce the regular representation $\widehat{N}_L$ of the generalized fusion algebra as the overlap matrices of the regular states.

4.2 General cases

We next turn to the case of twisted boundary states associated with a general modular invariant Z (see (2.9)) of the chiral algebra $\mathcal{A}$. In order to express twisted boundary states, we have to determine what kinds of twisted Ishibashi states are available in the modular invariant (2.9). As we have argued in Section 2.2, the twisted Ishibashi state $|(\lambda,\mu^*);\omega\rangle\rangle$ exists if and only if $\lambda=\omega(\mu)$. Accordingly, the set of labels for ω-twisted Ishibashi states, which we denote by $\mathcal{E}(\omega)$, takes the following form

$$
\mathcal{E}(\omega) = \{(\lambda,\mu^*)\,|\,(\lambda,\mu^*)\in\mathrm{Spec}(Z);\ \lambda=\omega(\mu)\}\,,
\tag{4.8}
$$

where Spec(Z) is the spectrum of bulk fields in Z (see (2.10)). A general boundary state $|A\rangle$ satisfying the boundary condition (2.29) is then written as

$$
|A\rangle = \sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega)}\Psi^{\omega}_{A(\lambda,\mu^*)}|(\lambda,\mu^*);\omega\rangle\rangle \quad (A\in\mathcal{V}^{\omega}\subset\hat{\mathcal{V}})\,.
\tag{4.9}
$$

Here $\mathcal{V}^{\omega}$ is the set of labels for the ω-twisted boundary states. We have also introduced the set $\hat{\mathcal{V}}$ of all the labels of twisted boundary states,

$$
\hat{\mathcal{V}} = \coprod_{\omega\in G}\mathcal{V}^{\omega}\,.
\tag{4.10}
$$

As in the case of the regular states, we assume that the number of twisted boundary states is equal to that of twisted Ishibashi states [3],

$$
|\mathcal{V}^{\omega}| = |\mathcal{E}(\omega)| \quad (\omega\in G)\,.
\tag{4.11}
$$

This condition together with the mutual consistency of boundary states implies that the boundary states labeled by $\mathcal{V}^\omega$ form a NIM-rep of the (ordinary) fusion algebra $\mathcal{F}(\mathcal{A})$ of $\mathcal{A}$ [2]. In particular, the matrix $(\Psi^\omega)_{A\,(\lambda,\mu^*)} = \Psi^\omega_{A(\lambda,\mu^*)}$ is unitary.

In the same way as the regular states (2.60), the overlap of two twisted boundary states can be expanded into a sum of the characters for representations of twisted chiral algebras,

$$\begin{aligned}
\langle B|\tilde{q}^{\frac{1}{2}H_c}|A\rangle &= \sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega_A)\cap\mathcal{E}(\omega_B)} \Psi^{\omega_A}_{A(\lambda,\mu^*)}\overline{\Psi^{\omega_B}_{B(\lambda,\mu^*)}}\frac{1}{S_{0\mu}}\chi_\mu^{\omega_B^{-1}\omega_A}(\tilde{q}) \\
&= \sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega_A)\cap\mathcal{E}(\omega_B)} \Psi^{\omega_A}_{A(\lambda,\mu^*)}\overline{\Psi^{\omega_B}_{B(\lambda,\mu^*)}}\frac{1}{S_{0\mu}} \sum_{N\in\mathcal{I}^{\omega_B^{-1}\omega_A}} \overline{S^{\omega_B^{-1}\omega_A}_{N\mu}}\chi_N(q) \\
&= \sum_{N\in\mathcal{I}^{\omega_A^{-1}\omega_B}} \sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega_A)\cap\mathcal{E}(\omega_B)} \Psi^{\omega_A}_{A(\lambda,\mu^*)}\frac{S^{\omega_A^{-1}\omega_B}_{N\mu}}{S_{0\mu}}\overline{\Psi^{\omega_B}_{B(\lambda,\mu^*)}}\,\chi_{N^*}(q) \\
&= \sum_{N\in\mathcal{I}^{\omega_A^{-1}\omega_B}} (\hat{n}_N)_A{}^B\,\chi_{N^*}(q)\,,
\end{aligned} \tag{4.12}$$

where ω_A is the automorphism type of $|A\rangle$ and we used the formula (2.38). A $|\hat{\mathcal{V}}|\times|\hat{\mathcal{V}}|$ matrix $\hat{n}_N$ is defined for each $N\in\hat{\mathcal{I}}$ as

$$(\hat{n}_N)_A{}^B = \sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega_A)\cap\mathcal{E}(\omega_B)} \Psi^{\omega_A}_{A(\lambda,\mu^*)}\frac{S^{\omega_N}_{N\mu}}{S_{0\mu}}\overline{\Psi^{\omega_B}_{B(\lambda,\mu^*)}}\times\delta_{\omega_A\omega_N,\omega_B}\,. \tag{4.13}$$

The entry of $\hat{n}_N$ should be non-negative integer for the mutual consistency of boundary states, since it represents the multiplicity of representation $N^*\in\hat{\mathcal{I}}$ in the open-string spectrum. In the rest of this section, we show that these matrices $\{\hat{n}_N|N\in\hat{\mathcal{I}}\}$ satisfy the generalized fusion algebra $\mathcal{F}(\mathcal{A};G)$.

We first rewrite the boundary states (4.9) in a form similar to the regular case (4.5). To do so, we have to generalize the basis (4.1) of Ishibashi states for $(\lambda,\lambda^*)\in\mathrm{Spec}(Z_c)$ to that for $(\lambda,\mu^*)\in\mathrm{Spec}(Z)$. In contrast with the case of the charge-conjugation invariant Z_c, the label (λ,μ^*) of bulk fields is in general not symmetric; there exists some $(\lambda,\mu^*)\in\mathrm{Spec}(Z)$ with $\lambda\neq\mu$. Although the untwisted Ishibashi state can not be constructed for (λ,μ^*) with $\lambda\neq\mu$, it is possible to obtain twisted states for (λ,μ^*) if $\lambda=\omega(\mu)$ for some $\omega\in G$. In other words, twisted Ishibashi states are available for (λ,μ^*) if λ belongs to the G-orbit $G\mu$ of μ,

$$G\mu = \{\omega(\mu)\,|\,\omega\in G\}\subset\mathcal{I}\,. \tag{4.14}$$

In the following, we consider only the case of $\lambda\in G\mu$, since the label (λ,μ^*) with $\lambda\notin G\mu$ does not appear in the twisted boundary states for the automorphism group G.

Let $\omega_{\lambda\mu}$ be an element of G that relates $\lambda\in G\mu$ with μ, namely $\lambda=\omega_{\lambda\mu}(\mu)$. In general, this element $\omega_{\lambda\mu}$ can not be determined uniquely due to the stabilizer $\mathcal{S}(\mu)$ of μ. Suppose that we have another element $\omega'_{\lambda\mu}\in G$ satisfying $\lambda=\omega'_{\lambda\mu}(\mu)$. Then $\omega_{\lambda\mu}^{-1}\omega'_{\lambda\mu}$ is an element of $\mathcal{S}(\mu)$ since $\omega_{\lambda\mu}^{-1}\omega'_{\lambda\mu}(\mu)=\omega_{\lambda\mu}^{-1}(\lambda)=\mu$. In other words, $\omega'_{\lambda\mu}$ belongs to the coset

$$\omega_{\lambda\mu}\mathcal{S}(\mu) = \{\omega_{\lambda\mu}\omega\,|\,\omega\in\mathcal{S}(\mu)\}\subset G\,. \tag{4.15}$$

This result shows that, for a given $(\lambda,\mu^*)\in\mathrm{Spec}(Z)$, the twisted Ishibashi state $|(\lambda,\mu^*);\omega\rangle\rangle$ exists if and only if $\omega\in\omega_{\lambda\mu}\mathcal{S}(\mu)$.

Based on this fact, one can generalize the basis (4.1) of twisted Ishibashi states in the following manner,

$$|((\lambda,\mu^*);\rho)_{ab}\rangle\rangle=\sqrt{\frac{\dim\rho}{|\mathcal{S}(\mu)|}}\sum_{\omega\in\mathcal{S}(\mu)}\overline{\rho^{ab}(\omega)}\,|(\lambda,\mu^*);\omega_{\lambda\mu}\omega\rangle\rangle\quad(\lambda=\omega_{\lambda\mu}(\mu);\ \rho\in\mathrm{Irr}(\mathcal{S}(\mu)))\,. \tag{4.16}$$

Similarly to eq.(4.2), one can express the original basis $|(\lambda,\mu^*);\omega\rangle\rangle$ in terms of $|((\lambda,\mu^*);\rho)_{ab}\rangle\rangle$,

$$|(\lambda,\mu^*);\omega\rangle\rangle=\frac{1}{\sqrt{|\mathcal{S}(\mu)|}}\sum_{\substack{\rho\in\mathrm{Irr}(\mathcal{S}(\mu))\\ 1\le a,b\le\dim\rho}}\sqrt{\dim\rho}\,\rho^{ab}(\omega_{\lambda\mu}^{-1}\omega)|((\lambda,\mu^*);\rho)_{ab}\rangle\rangle\quad(\omega\in\omega_{\lambda\mu}\mathcal{S}(\mu))\,. \tag{4.17}$$

The overlap of $|((\lambda,\mu^*);\rho)_{ab}\rangle\rangle$ with $|((\lambda,\mu^*);\rho')_{a'b'}\rangle\rangle$ can be calculated in exactly the same way as $|((\lambda,\lambda^*);\rho)_{ab}\rangle\rangle$,

$$\begin{aligned}
&\langle\langle((\lambda,\mu^*);\rho')_{a'b'}|\tilde{q}^{\frac{1}{2}H_c}|((\lambda,\mu^*);\rho)_{ab}\rangle\rangle\\
&\quad=\frac{\sqrt{\dim\rho'}\sqrt{\dim\rho}}{|\mathcal{S}(\mu)|}\sum_{\omega,\omega'\in\mathcal{S}(\mu)}\rho'^{a'b'}(\omega')\overline{\rho^{ab}(\omega)}\langle\langle(\lambda,\mu^*);\omega_{\lambda\mu}\omega'|\tilde{q}^{\frac{1}{2}H_c}|(\lambda,\mu^*);\omega_{\lambda\mu}\omega\rangle\rangle\\
&\quad=\frac{\sqrt{\dim\rho'}\sqrt{\dim\rho}}{|\mathcal{S}(\mu)|}\sum_{\omega,\omega'\in\mathcal{S}(\mu)}\rho'^{a'b'}(\omega')\overline{\rho^{ab}(\omega)}\frac{1}{S_{0\mu}}\chi_\mu^{\omega'^{-1}\omega}(\tilde{q})\\
&\quad=\frac{\sqrt{\dim\rho'}\sqrt{\dim\rho}}{|\mathcal{S}(\mu)|}\sum_{\omega,\omega'\in\mathcal{S}(\mu)}\rho'^{a'b'}(\omega')\overline{\rho^{ab}(\omega'\omega)}\frac{1}{S_{0\mu}}\chi_\mu^{\omega}(\tilde{q})\\
&\quad=\delta_{\rho\rho'}\delta^{aa'}\sum_{\omega\in\mathcal{S}(\mu)}\overline{\rho^{b'b}(\omega)}\frac{1}{S_{0\mu}}\chi_\mu^{\omega}(\tilde{q})\\
&\quad=\delta_{\rho\rho'}\delta^{aa'}\frac{1}{\hat{S}^{11}_{0(\mu;\rho)}}\chi^{b'b}_{(\mu;\rho)}(\tilde{q})\,.
\end{aligned} \tag{4.18}$$

The overlap between two states with different labels, $(\lambda',\mu'^*)\neq(\lambda,\mu^*)$, vanishes from the formula (2.38).

Substituting (4.17) in (4.9), the boundary state $|A\rangle$ can be brought to the following form,

$$\begin{aligned}
|A\rangle&=\sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega)}\Psi^{\omega_A}_{A(\lambda,\mu^*)}|(\lambda,\mu^*);\omega_A\rangle\rangle\\
&=\sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega)}\Psi^{\omega_A}_{A(\lambda,\mu^*)}\frac{1}{\sqrt{|\mathcal{S}(\mu)|}}\sum_{\substack{\rho\in\mathrm{Irr}(\mathcal{S}(\mu))\\ 1\le a,b\le\dim\rho}}\sqrt{\dim\rho}\,\rho^{ab}(\omega_{\lambda\mu}^{-1}\omega_A)|((\lambda,\mu^*);\rho)_{ab}\rangle\rangle\\
&=\sum_{((\lambda,\mu^*);\rho)\in\hat{\mathcal{E}}}\sum_{a,b}\hat{\Psi}^{ab}_{A((\lambda,\mu^*);\rho)}|((\lambda,\mu^*);\rho)_{ab}\rangle\rangle\,,
\end{aligned} \tag{4.19}$$

where we denote by $\hat{\mathcal{E}}$ the set of labels for the basis $|((\lambda,\mu^*);\rho)_{ab}\rangle\!\rangle$ available in the modular invariant (2.9),

$$\hat{\mathcal{E}} = \left\{ ((\lambda,\mu^*);\rho) \,\middle|\, (\lambda,\mu^*) \in \bigcup_{\omega\in G} \mathcal{E}(\omega);\; \rho \in \mathrm{Irr}(\mathcal{S}(\mu)) \right\} . \tag{4.20}$$

It should be noted that some (λ,μ^*) may appear more than once in $\hat{\mathcal{E}}$ corresponding to the multiple occurrence of bulk fields with representation (λ,μ^*) in the modular invariant (2.9). The boundary state coefficient $\hat{\Psi}^{ab}_{A((\lambda,\mu^*);\rho)}$ is related to the original coefficient $\Psi^{\omega_A}_{A(\lambda,\mu^*)}$ as follows

$$\hat{\Psi}^{ab}_{A((\lambda,\mu^*);\rho)} = \begin{cases} \Psi^{\omega_A}_{A(\lambda,\mu^*)} \sqrt{\frac{\dim\rho}{|\mathcal{S}(\mu)|}}\rho^{ab}(\omega^{-1}_{\lambda\mu}\omega_A) & ((\lambda,\mu^*)\in\mathcal{E}(\omega_A),\ \rho\in\mathrm{Irr}(\mathcal{S}(\mu)))\,, \\ 0 & ((\lambda,\mu^*)\notin\mathcal{E}(\omega_A))\,. \end{cases} \tag{4.21}$$

This form is completely parallel to the generalized S-matrix (3.11). Actually, from the unitarity of Ψ^ω for all $\omega\in G$, one can prove that the matrix $\hat{\Psi}$ is also unitary,

$$|\hat{\mathcal{V}}| = \sum_{\omega\in G}|\mathcal{V}^\omega| = \sum_{\omega\in G}|\mathcal{E}(\omega)| = \sum_{(\lambda,\mu^*)\in\bigcup_{\omega\in G}\mathcal{E}(\omega)} |\mathcal{S}(\mu)| = \sum_{((\lambda,\mu^*);\rho)\in\hat{\mathcal{E}}} (\dim\rho)^2\,, \tag{4.22}$$

$$\begin{aligned} &\sum_{((\lambda,\mu^*);\rho)\in\hat{\mathcal{E}};a,b} \hat{\Psi}^{ab}_{A((\lambda,\mu^*);\rho)} \overline{\hat{\Psi}^{ab}_{B((\lambda,\mu^*);\rho)}} \\ &\quad= \sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega_A)\cap\mathcal{E}(\omega_B)} \Psi^{\omega_A}_{A(\lambda,\mu^*)} \overline{\Psi^{\omega_B}_{B(\lambda,\mu^*)}} \sum_{\rho\in\mathrm{Irr}(\mathcal{S}(\mu))} \sum_{a,b} \frac{\dim\rho}{|\mathcal{S}(\mu)|}\rho^{ab}(\omega^{-1}_{\lambda\mu}\omega_A)\overline{\rho^{ab}(\omega^{-1}_{\lambda\mu}\omega_B)} \\ &\quad= \sum_{(\lambda,\mu^*)\in\mathcal{E}(\omega_A)} \Psi^{\omega_A}_{A(\lambda,\mu^*)} \overline{\Psi^{\omega_A}_{B(\lambda,\mu^*)}} \times \delta_{\omega_A\omega_B} \\ &\quad= \delta_{AB}\,, \end{aligned} \tag{4.23}$$

where we used the fact that $\omega^{-1}_{\lambda\mu}\omega_A = \omega^{-1}_{\lambda\mu}\omega_B$ in $\mathcal{S}(\mu)\subset G$ implies $\omega_A=\omega_B$ in G.

Using the form given in (4.19) together with the formula (4.18), one can calculate the overlap of two boundary states $|A\rangle$ and $|B\rangle$ in the following way,

$$\begin{aligned} \langle B|\tilde{q}^{\frac{1}{2}H_c}|A\rangle &= \sum_{((\lambda,\mu^*);\rho)\in\hat{\mathcal{E}}} \sum_{a,b}\sum_{a',b'} \overline{\hat{\Psi}^{a'b'}_{B((\lambda,\mu^*);\rho)}} \hat{\Psi}^{ab}_{A((\lambda,\mu^*);\rho)}\, \delta^{aa'} \frac{1}{\hat{S}^{11}_{0(\mu;\rho)}} \chi^{b'b}_{(\mu;\rho)}(\tilde{q}) \\ &= \sum_{((\lambda,\mu^*);\rho)\in\hat{\mathcal{E}}} \sum_{a,b}\sum_{b'} \overline{\hat{\Psi}^{ab'}_{B((\lambda,\mu^*);\rho)}} \hat{\Psi}^{ab}_{A((\lambda,\mu^*);\rho)} \frac{1}{\hat{S}^{11}_{0(\mu;\rho)}} \sum_{N\in\hat{\mathcal{I}}} \overline{\hat{S}^{b'b}_{N(\mu;\rho)}}\,\chi_N(q) \\ &= \sum_{((\lambda,\mu^*);\rho)\in\hat{\mathcal{E}}} \sum_{a,b}\sum_{b'} \overline{\hat{\Psi}^{ab'}_{B((\lambda,\mu^*);\rho)}} \hat{\Psi}^{ab}_{A((\lambda,\mu^*);\rho)} \sum_{N\in\hat{\mathcal{I}}} \frac{\hat{S}^{bb'}_{N^*(\mu;\rho)}}{\hat{S}^{11}_{0(\mu;\rho)}}\,\chi_N(q) \\ &= \sum_{N\in\hat{\mathcal{I}}}\sum_{((\lambda,\mu^*);\rho)\in\hat{\mathcal{E}}} \sum_{a,b}\sum_{a'b'} \hat{\Psi}^{ab}_{A((\lambda,\mu^*);\rho)}\,\delta^{aa'}\,\mathfrak{b}^{bb'}_{(\mu;\rho)}(N^*)\overline{\hat{\Psi}^{a'b'}_{B((\lambda,\mu^*);\rho)}}\,\chi_N(q)\,. \end{aligned} \tag{4.24}$$

Comparing this with eq.(4.12), we obtain the following expression for the overlap matrix $\hat{n}_N$,

$$\hat{n}_N = \hat{\Psi}\left[\bigoplus_{((\lambda,\mu^*);\rho)\in\hat{\mathcal{E}}} \big(\underbrace{\mathfrak{b}_{(\mu;\rho)}(N)\oplus\cdots\oplus\mathfrak{b}_{(\mu;\rho)}(N)}_{(\dim\rho)\text{ terms}}\big)\right]\hat{\Psi}^\dagger\,. \tag{4.25}$$

This has exactly the same structure as eq.(3.20) for the regular representation matrix $\widehat{N}_N$, from which we have shown $\widehat{N}_N$ satisfies the generalized fusion algebra. We can repeat the same thing here since the matrix $\hat{\Psi}$ is unitary from the assumption (4.11) of completeness. For a unitary $\hat{\Psi}$, the above expression (4.25) for the overlap matrix $\hat{n}_N$ means that $\hat{n}_N$ is similar to a direct sum of $\mathfrak{b}_{(\mu;\rho)}(N)$. Since $\mathfrak{b}_{(\mu;\rho)}(N)$ satisfies the generalized fusion algebra for all $(\mu;\rho)\in\hat{\mathcal{I}}^*$, the matrix $\hat{n}_N$ also satisfies the generalized fusion algebra. The mutual consistency of twisted boundary states requires that the coefficients $(\hat{n}_N)_A{}^B$ of χ_{N*} in the cylinder amplitude take values in non-negative integers for all $N\in\hat{\mathcal{I}}$. Hence we are led to the conclusion that the overlap matrix $\hat{n}_N$ for a set of mutually consistent boundary states realizes a NIM-rep of the generalized fusion algebra $\mathcal{F}(\mathcal{A};G)$ if the condition (4.11) is satisfied.

5 Examples

As we have shown in the previous section, a consistent set of twisted boundary states in any modular invariant form a NIM-rep of the generalized fusion algebra. In this section, we check this for three concrete chiral algebras, $u(1)_k$, $su(3)_k$ and $su(3)_1^{\oplus 3}$, by explicitly constructing twisted boundary states for each case.

5.1 $u(1)_k$

The simplest chiral algebra with non-trivial automorphisms is $u(1)_k$, where the level k is a positive integer. There are $2k$ irreducible representations labeled by positive integer n mod $2k$,

$$\mathcal{I}=\{(n)\,|\,n=0,1,\ldots,2k-1\}\,, \tag{5.1}$$

for which the modular transformation matrix takes the form

$$S_{mn}=\frac{1}{\sqrt{2k}}e^{-\frac{\pi i}{k}mn}\,. \tag{5.2}$$

Therefore, for the charge-conjugation modular invariant, we have $2k$ untwisted boundary states,

$$|m\rangle=\sum_{n\in\mathcal{I}}S_{mn}|(n,n^*)\rangle\!\rangle \quad (m\in\mathcal{I})\,. \tag{5.3}$$

The charge-conjugation ω_c is an automorphism of $u(1)_k$, which acts on $\mathcal{I}$ as

$$\omega_c\,:\,(n)\mapsto(n^*)=(-n)=(2k-n)\quad(n\in\mathcal{I})\,. \tag{5.4}$$

There are two representations fixed by ω_c,

$$\mathcal{I}(\omega_c) = \{(0),\, (k)\}\,. \tag{5.5}$$

Corresponding to this, we have two twisted boundary states, which we denote by $|\pm\rangle$,

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|(0,0);\omega_c\rangle\rangle \pm |(k,k);\omega_c\rangle\rangle)\,. \tag{5.6}$$

The overlap of $|\pm\rangle$ with the untwisted state $|0\rangle$ yields the character $\chi_\pm$ of the twisted chiral algebra $u(1)_k^{\omega_c}$ (see eq.(2.46)),

$$\chi_\pm(q) = \langle 0|\tilde{q}^{\frac{1}{2}H_c}|\pm\rangle = \frac{1}{\sqrt{2}}(\chi_0^{\omega_c}(\tilde{q}) \pm \chi_k^{\omega_c}(\tilde{q}))\,, \tag{5.7}$$

where $\chi_n^{\omega_c}$ is the twining character of (n) for ω_c. From this equation, one obtains the modular transformation matrix S^{ω_c}

$$S^{\omega_c} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\,, \tag{5.8}$$

where the rows and the columns are ordered as $\mathcal{I}^{\omega_c} = \{+,-\}$ and $\mathcal{I}(\omega_c) = \{0,k\}$, respectively. Since $\omega_c^{-1} = \omega_c$ and S^{ω_c} is real, the relation (2.59) implies that the twisted representations $(\pm) \in \mathcal{I}^{\omega_c}$ are self-conjugate, $(\pm^*) = (\pm)$.

Using these data and the formula (see eq.(2.60))

$$\langle L^*|\tilde{q}^{\frac{1}{2}H_c}|M\rangle = (L)\times(M) = \sum_{N\in\hat{\mathcal{I}}} \widehat{\mathcal{N}}_{LM}{}^{N}\chi_N(q) \quad (L,M\in\hat{\mathcal{I}})\,, \tag{5.9}$$

one can determine the generalized fusion coefficients $\widehat{\mathcal{N}}_{LM}{}^{N}$ for the chiral algebra $u(1)_k$ and the automorphism group $G_c = \{1,\omega_c\} \cong \mathbb{Z}_2$. The result is as follows,

$$\begin{aligned}
&(l)\times(m) = (l+m)\,,\\
&(l)\times(\pm) = (\pm)\times(l) = \begin{cases}(\pm) & (l=0 \mod 2)\\ (\mp) & (l=1 \mod 2)\end{cases}\,,\\
&(+)\times(+) = (-)\times(-) = (0)+(2)+\cdots+(2k-2)\,,\\
&(+)\times(-) = (-)\times(+) = (1)+(3)+\cdots+(2k-1)\,,
\end{aligned} \tag{5.10}$$

where $(l),(m)\in\mathcal{I}$ and $(\pm)\in\mathcal{I}^{\omega_c}$. This defines the generalized fusion algebra $\mathcal{F}(u(1)_k;G_c)$ for $G_c = \{1,\omega_c\}\cong\mathbb{Z}_2$. As is easily seen, this algebra is generated by (1) and $(+)$. Hence $\mathcal{F}(u(1)_k;G_c)$ is an extension of the ordinary fusion algebra $\mathcal{F}(u(1)_k)$, which is the group algebra of $\mathbb{Z}_{2k}$, by the twisted representation $(+)$. Note that $\mathcal{F}(u(1)_k;G_c)$ is commutative, $(L)\times(M) = (M)\times(L)$, reflecting the fact that the automorphism group $G_c\cong\mathbb{Z}_2$ is abelian.

We next turn to the case of modular invariants other than the charge-conjugation one and check that the associated boundary states form a NIM-rep of the generalized fusion algebra

(5.10). We consider two cases: a simple current extension of $k = 4$ and a non-diagonal invariant of $k = 6$.

<u>$k = 4$</u>

For $k = 4$, we have the following block-diagonal invariant

$$Z = |\chi_0 + \chi_4|^2 + |\chi_2 + \chi_6|^2 \,, \tag{5.11}$$

which is a simple current extension [16] of $u(1)_4$ by $(4) \in \mathcal{I}$. The chiral algebra of the extended theory is $u(1)_1$ and each block corresponds to an irreducible representation of $u(1)_1$. Since there are two irreducible representations for $u(1)_1$, we have two untwisted boundary states that keep $u(1)_1$ (see eq.(5.3))

$$\frac{1}{\sqrt{2}}(|(0',0')\rangle\rangle \pm |(1',1')\rangle\rangle) \,, \tag{5.12}$$

where we distinguish representations of $u(1)_1$ from those of $u(1)_4$ by putting a prime. Since $u(1)_1$ contains $u(1)_4$ as a subalgebra, we can regard these states as untwisted boundary states of $u(1)_4$ associated with the modular invariant (5.11). From the branching rule

$$(0') = (0) \oplus (4) \,, \quad (1') = (2) \oplus (6) \,, \tag{5.13}$$

and our normalization (2.7) for Ishibashi states, we can express Ishibashi states of $u(1)_1$ by those of $u(1)_4$,

$$|(0',0')\rangle\rangle = \frac{1}{\sqrt{2}}(|(0,0)\rangle\rangle + |(4,4)\rangle\rangle) \,, \quad |(1',1')\rangle\rangle = \frac{1}{\sqrt{2}}(|(2,2^*)\rangle\rangle + |(6,6^*)\rangle\rangle) \,. \tag{5.14}$$

Substituting this into eq.(5.12), we obtain two untwisted boundary states of $u(1)_4$,

$$\begin{aligned} |\tilde{0}\rangle &= \frac{1}{2}(|(0,0)\rangle\rangle + |(4,4)\rangle\rangle) + \frac{1}{2}(|(2,2^*)\rangle\rangle + |(6,6^*)\rangle\rangle) \,, \\ |\tilde{2}\rangle &= \frac{1}{2}(|(0,0)\rangle\rangle + |(4,4)\rangle\rangle) - \frac{1}{2}(|(2,2^*)\rangle\rangle + |(6,6^*)\rangle\rangle) \,. \end{aligned} \tag{5.15}$$

Since there are four untwisted Ishibashi states in (5.11),

$$\mathcal{E}(1) = \{(0,0), (2,2^*), (4,4), (6,6^*)\} \,, \tag{5.16}$$

we need two more boundary states for completeness. The remaining two can be constructed by, *e.g.*, the fusion with representations of the unextended chiral algebra [26, 2, 27]. (See [27] for a detailed exposition of this procedure.) The result is as follows,

$$\begin{aligned} |\tilde{1}\rangle &= \frac{1}{2}(|(0,0)\rangle\rangle - |(4,4)\rangle\rangle) - \frac{i}{2}(|(2,2^*)\rangle\rangle - |(6,6^*)\rangle\rangle) \,, \\ |\tilde{3}\rangle &= \frac{1}{2}(|(0,0)\rangle\rangle - |(4,4)\rangle\rangle) + \frac{i}{2}(|(2,2^*)\rangle\rangle - |(6,6^*)\rangle\rangle) \,. \end{aligned} \tag{5.17}$$

Consequently, the boundary state coefficients Ψ for untwisted boundary states associated with (5.11) take the following form

$$\Psi = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}, \tag{5.18}$$

where the column is ordered as in (5.16).

The restriction of the charge-conjugation of $u(1)_1$ to the subalgebra $u(1)_4$ is clearly the charge-conjugation of $u(1)_4$. In other words, the charge-conjugation of $u(1)_4$ has a lift to the extended chiral algebra $u(1)_1$. (The lifting of automorphisms for RCFTs is studied in [28].) This enables us to construct twisted boundary states of $u(1)_4$ starting from those of $u(1)_1$. Since two representations of $u(1)_1$ are fixed by the charge-conjugation, we have two twisted boundary states of $u(1)_1$,

$$\frac{1}{\sqrt{2}}(|(0',0');\omega_c\rangle\!\rangle \pm |(1',1');\omega_c\rangle\!\rangle)\,. \tag{5.19}$$

This is completely the same form as the untwisted states. Hence one can proceed in the same way as the untwisted states to obtain four twisted states of $u(1)_4$. The boundary state coefficients Ψ^{ω_c} of twisted states coincide with Ψ given in (5.18) with the spectrum

$$\mathcal{E}(\omega_c) = \{(0,0),(6,2^*),(4,4),(2,6^*)\} \tag{5.20}$$

instead of $\mathcal{E}(1)$.

From these data, one can calculate the mutual overlap of boundary states and the associated open string spectrum $\hat{n}$ defined in (4.13). Since we have $4+4=8$ boundary states, $\hat{n}$ is a 8×8 matrix and takes the form

$$\hat{n}_l = \begin{pmatrix} (n_1^{(4)})^l & O \\ O & (n_1^{(4)})^l \end{pmatrix} \; (l=0,1,\ldots,7), \quad \hat{n}_\pm = \begin{pmatrix} O & n_\pm \\ n_\pm & O \end{pmatrix}, \tag{5.21}$$

where O is a 4×4 matrix with all entries being 0 and $n_{1,\pm}$ are defined as

$$n_1^{(4)} = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix}, \; n_+ = \begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \end{pmatrix}, \; n_- = \begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{pmatrix}. \tag{5.22}$$

One can confirm that these 10 matrices actually satisfy the generalized fusion algebra (5.10) for $k=4$, namely,

$$(\hat{n}_1)^8 = 1\,, \quad \hat{n}_1\hat{n}_+ = \hat{n}_-\,, \quad \hat{n}_+\hat{n}_+ = \hat{n}_0 + \hat{n}_2 + \hat{n}_4 + \hat{n}_6\,. \tag{5.23}$$

The boundary states associated with the simple current extension (5.10) therefore realize a NIM-rep of the generalized fusion algebra $\mathcal{F}(u(1)_4; G_c)$. Note that this 8-dimensional NIM-rep is distinct from the regular one, which is $8+2=10$ dimensional. Hence we have obtained

two NIM-reps of $\mathcal{F}(u(1)_4; G_c)$; one of them corresponds to the charge-conjugation invariant whereas the other originates from the block diagonal invariant (5.10).

Since the generalized fusion algebra contains the ordinary fusion algebra as a subalgebra, one can decompose a NIM-rep of the former into NIM-reps of the latter. Conversely, several NIM-reps (more precisely, $|G|$ NIM-reps) of the ordinary fusion algebra may be combined to yield a NIM-rep of the generalized fusion algebra. For $u(1)_4$, there are four NIM-reps of the ordinary fusion algebra $\mathcal{F}(u(1)_4)$ corresponding to four subgroups of the simple current group $\mathbb{Z}_8$ [7]. Two of them, the regular NIM-rep and the two-dimensional one, form the regular NIM-rep of the generalized fusion algebra. The four-dimensional one (5.18) is paired with itself to form a 8-dimensional NIM-rep of the generalized fusion algebra, as we have seen above. The remaining one, which is one-dimensional and unphysical, is also paired with itself to yield a two-dimensional NIM-rep of the generalized fusion algebra. Consequently, there are three NIM-reps for the generalized fusion algebra $\mathcal{F}(u(1)_4; G_c)$; two of them are physical and correspond to the charge-conjugation modular invariant and the simple current extension (5.10), respectively, while the remaining one is unphysical and has no associated modular invariant.

<u>$k = 6$</u>

For $k = 6$, we have the following non-diagonal invariant,

$$\begin{aligned} Z = &|\chi_0|^2 + |\chi_2|^2 + |\chi_4|^2 + |\chi_6|^2 + |\chi_8|^2 + |\chi_{10}|^2 \\ &+ \chi_1\overline{\chi_7} + \chi_3\overline{\chi_9} + \chi_5\overline{\chi_{11}} + \chi_7\overline{\chi_1} + \chi_9\overline{\chi_3} + \chi_{11}\overline{\chi_5}\,, \end{aligned} \tag{5.24}$$

which is the $\mathbb{Z}_2$-orbifold of $u(1)_6$. Let $|m\rangle$ $(m \in \mathcal{I} = \{0, 1, 2, \cdots, 11\})$ be the untwisted boundary state in the charge-conjugation modular invariant of $u(1)_6$ defined in (5.3). The orbifold action on $|m\rangle$ reads $|m\rangle \mapsto |m{+}6\rangle$, and the untwisted state in (5.24) can be obtained by averaging $|m\rangle$ over the $\mathbb{Z}_2$-orbit,

$$|m\rangle_{\mathbb{Z}_2} = \frac{1}{\sqrt{2}}(|m\rangle + |m+6\rangle) = \frac{1}{\sqrt{6}} \sum_{n=0,1,\ldots,5} e^{-\frac{\pi i}{3}mn} |(2n, 2n^*)\rangle\!\rangle\,. \tag{5.25}$$

The ω_c-twisted boundary states in (5.24) can be constructed as the untwisted states in the $\mathbb{Z}_3$-orbifold of $u(1)_6$, since the $\mathbb{Z}_3$-orbifold is the T-dual of (5.24). The untwisted states in the $\mathbb{Z}_3$-orbifold are obtained by averaging $|m\rangle$ over the $\mathbb{Z}_3$-orbit,

$$|m\rangle_{\mathbb{Z}_3} = \frac{1}{\sqrt{3}}(|m\rangle + |m+4\rangle + |m+8\rangle) = \frac{1}{2} \sum_{n=0,1,2,3} e^{-\frac{\pi i}{2}mn} |(3n, 3n^*)\rangle\!\rangle\,. \tag{5.26}$$

The ω_c-twisted states in the $\mathbb{Z}_2$-orbifold (5.24), which we denote by $|m;\omega_c\rangle_{\mathbb{Z}_2}$, are constructed by acting ω_c on the holomorphic sector of $|m\rangle_{\mathbb{Z}_3}$,

$$|m;\omega_c\rangle_{\mathbb{Z}_2} = R(\omega_c)|m\rangle_{\mathbb{Z}_3} = \frac{1}{2} \sum_{n=0,1,2,3} e^{-\frac{\pi i}{2}mn} |(12-3n, 3n^*);\omega_c\rangle\!\rangle\,. \tag{5.27}$$

In this way, we obtain 6 untwisted states and 4 twisted states in (5.24). The overlap matrix $\hat{n}$ is a 10×10 matrix and takes the form,

$$\hat{n}_l = \begin{pmatrix} (n_1^{(6)})^l & O^{(6,4)} \\ O^{(4,6)} & (n_1^{(4)})^l \end{pmatrix} \; (l = 0, 1, \ldots, 11), \quad \hat{n}_\pm = \begin{pmatrix} O^{(6,6)} & n_\pm \\ n_\pm^T & O^{(4,4)} \end{pmatrix}, \tag{5.28}$$

where $n_1^{(4)}$ is defined in (5.22), $O^{(m,n)}$ is a $m \times n$ zero matrix and the other matrices are defined as

$$n_1^{(6)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; n_+ = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \; n_- = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{5.29}$$

One can easily check that these matrices satisfy the generalized fusion algebra (5.10) for $k = 6$.

The simple current group of $u(1)_6$ is isomorphic to $\mathbb{Z}_{12}$. Hence we have 6 NIM-reps of the ordinary fusion algebra $\mathcal{F}(u(1)_6)$ corresponding to 6 divisors $\{1, 2, 3, 4, 6, 12\}$ of 12 [7]. In the same way as the case of $k = 4$, two of them, the regular and the two-dimensional ones, form the regular NIM-rep of the generalized fusion algebra $\mathcal{F}(u(1)_6; G_c)$. NIM-reps of dimension 6 and 4 are combined to yield another NIM-rep of $\mathcal{F}(u(1)_6; G_c)$ associated with the $\mathbb{Z}_2$-orbifold (or equivalently, the $\mathbb{Z}_3$-orbifold) of $u(1)_6$, as we have shown above. In contrast with the case of $k = 4$, however, the remaining two NIM-reps of dimension 1 and 3 can not form a NIM-rep of $\mathcal{F}(u(1)_6; G_c)$. Actually, one can show that the overlap matrices $\hat{n}$ containing the one-dimensional NIM-rep of $\mathcal{F}(u(1)_6)$ as a factor and satisfying the generalized fusion algebra $\mathcal{F}(u(1)_6; G_c)$ necessarily have non-integral entries and can not be a NIM-rep. The case of the three-dimensional NIM-rep is shown in the same way. Consequently, we have two NIM-reps of $\mathcal{F}(u(1)_6; G_c)$, one of which corresponds to the charge-conjugation (or the diagonal) invariant while the other is associated with the $\mathbb{Z}_2$-orbifold (or its T-dual) of $u(1)_6$.

5.2 $su(3)_k$

Our next example is $su(3)_k$ with the charge-conjugation ω_c as an automorphism of the chiral algebra. Since the automorphism group $G_c = \{1, \omega_c\}$ consists of two elements, we can expect that two NIM-reps of the ordinary fusion algebra $\mathcal{F}(su(3)_k)$ are combined to yield a NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_k; G_c)$. We check this for the case of $k = 3$ and $k = 5$, in which there are respectively four and six independent NIM-reps of $\mathcal{F}(su(3)_k)$. We present here only the outline of our analysis and give the details in Appendix B.

<u>$k = 3$</u>

For $k = 3$, there are four NIM-reps ($A^{(6)}, A^{(6)*}, D^{(6)}$ and $D^{(6)*}$ [8]) of the ordinary fusion algebra $\mathcal{F}(su(3)_3)$ [19, 2]. Two of them, $A^{(6)}$ and $A^{(6)*}$, correspond to respectively the

[8] The superscript stands for the sum $k + h^\vee = k + 3$, where $h^\vee$ is the dual Coxeter number of $su(3)$.

untwisted and the twisted boundary states in the charge-conjugation modular invariant and constitute the regular NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_3; G_c)$. The remaining two, $D^{(6)}$ and $D^{(6)*}$, express respectively the untwisted and the twisted boundary states corresponding to the following block diagonal modular invariant,

$$Z_{D^{(6)}} = |\chi_{(0,0)} + \chi_{(3,0)} + \chi_{(0,3)}|^2 + 3|\chi_{(1,1)}|^2 \,, \tag{5.30}$$

which is a simple current extension of $su(3)_3$ by $(3,0)$. Here the subscripts of characters denote the Dynkin label of $su(3)$. The extended chiral algebra of this invariant is $so(8)_1$ and one can construct the boundary states associated with (5.30) using the data of $so(8)_1$. Actually, as is shown in Appendix A, the automorphism group G_c has a lift $\tilde{G}_c$ to $so(8)$, which is isomorphic to the symmetric group S_3. Accordingly, the boundary states for $D^{(6)}$ and $D^{(6)*}$ are obtained by the $\tilde{G}_c$-twisted states in $so(8)$. By an explicit calculation of the overlap of the twisted states with the untwisted ones in $su(3)$, we can show that two NIM-reps, $D^{(6)}$ and $D^{(6)*}$, of $\mathcal{F}(su(3)_3)$ form a NIM-rep of $\mathcal{F}(su(3)_3; G_c)$. In this way, four NIM-reps of the ordinary fusion algebra $\mathcal{F}(su(3)_3)$ are organized into two NIM-reps of the generalized fusion algebra $\mathcal{F}(su(3)_3; G_c)$; one corresponds to the charge-conjugation modular invariant while the other is originated from the simple current extension (5.30).

$\underline{k=5}$

For $k = 5$, there are six NIM-reps ($A^{(8)}, A^{(8)*}, D^{(8)}, D^{(8)*}, E^{(8)}$ and $E^{(8)*}$) of the ordinary fusion algebra $\mathcal{F}(su(3)_5)$ [19, 2]. Similarly to the case of $k = 3$, $A^{(8)}$ and $A^{(8)*}$ express the boundary states in the charge-conjugation invariant and form the regular NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_5; G_c)$. The remaining four correspond to other modular invariants. Two of them, $D^{(8)}$ and $D^{(8)*}$, represent the boundary states in the simple current invariant

$$\begin{aligned} Z_{D^{(8)}} = {} & |\chi_{(0,0)}|^2 + |\chi_{(1,1)}|^2 + |\chi_{(2,2)}|^2 + |\chi_{(3,0)}|^2 + |\chi_{(0,3)}|^2 + |\chi_{(4,1)}|^2 + |\chi_{(1,4)}|^2 \\ & + \big(\chi_{(5,0)}\overline{\chi_{(0,5)}} + \chi_{(3,1)}\overline{\chi_{(1,3)}} + \chi_{(1,2)}\overline{\chi_{(2,1)}} \\ & \quad + \chi_{(2,3)}\overline{\chi_{(0,2)}} + \chi_{(2,0)}\overline{\chi_{(3,2)}} + \chi_{(0,4)}\overline{\chi_{(1,0)}} + \chi_{(0,1)}\overline{\chi_{(4,0)}} + \text{ c.c}\big) \,, \end{aligned} \tag{5.31}$$

while $E^{(8)}$ and $E^{(8)*}$ correspond to the exceptional invariant

$$\begin{aligned} Z_{E^{(8)}} = {} & |\chi_{(0,0)} + \chi_{(2,2)}|^2 + |\chi_{(0,2)} + \chi_{(3,2)}|^2 + |\chi_{(1,2)} + \chi_{(5,0)}|^2 \\ & + |\chi_{(3,0)} + \chi_{(0,3)}|^2 + |\chi_{(2,1)} + \chi_{(0,5)}|^2 + |\chi_{(2,0)} + \chi_{(2,3)}|^2 \,, \end{aligned} \tag{5.32}$$

which originates from the conformal embedding $su(3)_5 \subset su(6)_1$.

The invariant (5.31) is the $\mathbb{Z}_3$-orbifold of $su(3)_5$ and we can obtain untwisted states by averaging the $\mathbb{Z}_3$-orbit of untwisted states in the charge-conjugation invariant. On the other hand, the twisted states in the charge-conjugation invariant are fixed points of the orbifold action and we need an appropriate resolution of them by the Ishibashi states from the twisted sector of the orbifold to obtain twisted states in (5.31). For the invariant (5.32), one can construct twisted boundary states from the data of $su(6)$, since the charge-conjugation of $su(3)$ is lifted to that of $su(6)$. One obtains six untwisted states together with two twisted

states using the data of $su(6)$. The remaining states of $E^{(8)}$ and $E^{(8)*}$ can be obtained by, *e.g.*, the fusion with representations of $su(3)$.

From the explicit form of boundary states in (5.31) and (5.32), we can calculate the overlap of twisted states with untwisted ones to show that two NIM reps of $\mathcal{F}(su(3)_5)$ corresponding to the same modular invariant form a NIM-rep of $\mathcal{F}(su(3)_5; G_c)$. Consequently, we obtain three NIM-reps of the generalized fusion algebra $\mathcal{F}(su(3)_5; G_c)$ corresponding to three modular invariants available for $k = 5$.

5.3 $su(3)_1^{\oplus 3}$

The final example we consider is $su(3)_1^{\oplus 3} = su(3)_1 \oplus su(3)_1 \oplus su(3)_1$. The details are given in Appendix C. This chiral algebra has the automorphism group consisting of all the permutations of three factors, which is isomorphic to S_3, and we obtain a non-commutative generalized fusion algebra $\mathcal{F}(su(3)_1^{\oplus 3}; S_3)$. For the chiral algebra $su(3)_1^{\oplus 3}$, we have a non-trivial block-diagonal modular invariant

$$\begin{aligned} Z_{E_6} = &\left|\chi_{(0,0,0)} + \chi_{(1,1,1)} + \chi_{(2,2,2)}\right|^2 \\ &+ \left|\chi_{(0,2,1)} + \chi_{(1,0,2)} + \chi_{(2,1,0)}\right|^2 + \left|\chi_{(0,1,2)} + \chi_{(2,0,1)} + \chi_{(1,2,0)}\right|^2 , \end{aligned} \tag{5.33}$$

where the subscripts stands for the label of representations of $su(3)_1^{\oplus 3}$ and the numbers $0, 1, 2$ correspond respectively to the fundamental weights $\Lambda_0, \Lambda_1, \Lambda_2$ of $su(3)_1$. This invariant originates from the conformal embedding $su(3)_1^{\oplus 3} \subset E_{6,1}$. [9] Since the automorphism group S_3 of $su(3)_1^{\oplus 3}$ has a lift to $E_{6,1}$, we can construct twisted boundary states for S_3 [11, 29] in this invariant. In the same way as the case of $su(3)_k$, we can calculate the overlap of twisted states to check that the overlap matrices indeed form a NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_1^{\oplus 3}; S_3)$.

6 Graph fusion algebras

Given a boundary state coefficient matrix $\Psi_{\alpha\lambda}(\alpha \in \mathcal{V}, \lambda \in \mathcal{E})$ [10] and $0 \in \mathcal{V}$ such that $\Psi_{0\lambda} \neq 0$, one can define the graph fusion algebra [19, 20, 2]

$$(\alpha) \times (\beta) = \sum_{\gamma \in \mathcal{V}} G_{\alpha\beta}{}^{\gamma}\, (\gamma) \qquad (\alpha, \beta \in \mathcal{V}) \tag{6.1}$$

with the structure constant

$$G_{\alpha\beta}{}^{\gamma} = \sum_{\lambda \in \mathcal{E}} \frac{\Psi_{\alpha\lambda}\Psi_{\beta\lambda}\overline{\Psi_{\gamma\lambda}}}{\Psi_{0\lambda}} \,. \tag{6.2}$$

Since this has the same form as the Verlinde formula (3.1), one can consider the graph fusion algebra as a generalization of ordinary fusion algebras. Actually, for the case of boundary states associated with a block diagonal invariant, it is observed [19] that the graph fusion algebra contains the fusion algebra of the extended theory as a subalgebra.

[9] $E_{6,1}$ means E_6 at level 1.

[10] In this section, we omit the label for the anti-holomorphic sector of Ishibashi states for simplicity.

For the case of simple current extensions, this relation of graph fusion algebras and the fusion algebra of the extended theory can be explained from the point of view of the generalized fusion algebra. As an illustration, we consider the D-type modular invariant of $su(2)_4$,

$$Z = |\chi_0 + \chi_4|^2 + 2|\chi_2|^2 \,, \tag{6.3}$$

where the subscript stands for the Dynkin label of $su(2)$. This invariant is obtained from a simple current extension of $su(2)_4$ by the simple current group $\Gamma = \{(0),(4)\} \cong \mathbb{Z}_2$. The extended chiral algebra is $su(3)_1$ and one can regard (6.3) as the charge-conjugation invariant of $su(3)_1$

$$Z = |\chi_{(0,0)}|^2 + |\chi_{(1,0)}|^2 + |\chi_{(0,1)}|^2 \,, \tag{6.4}$$

where representations are again denoted by their Dynkin labels. One obtains (6.3) from (6.4) using the branching rule for the representations of $su(3)_1$

$$(0,0) = (0) \oplus (4) \,, \quad (1,0) = (2) \,, \quad (0,1) = (2) \,. \tag{6.5}$$

The unextended chiral algebra in simple current extensions can be characterized by the fixed point algebra of a certain automorphism group G of the extended chiral algebra [8], where G is the character group for the simple current group Γ used in the extension and hence isomorphic to Γ; $G = \Gamma^* \cong \Gamma$. For the case of $su(2)_4 \subset su(3)_1$, G is nothing but the charge-conjugation automorphism group G_c of $su(3)_1$, since $su(2)_4$ is the fixed point algebra of the charge-conjugation ω_c. The eigenvalue of ω_c is ± 1, since ω_c has order 2. The vacuum representation of $su(3)_1$ is therefore decomposed into two parts according to the eigenvalue of ω_c. Since ω_c fixes $su(2)_4 \subset su(3)_1$, this decomposition of $(0,0)$ coincides with the decomposition (6.5) into representations of $su(2)_4$. The representation (0) is $su(2)_4$ itself and corresponds to $\omega_c = 1$. The representation (4) contains the highest root of $su(3)$, whose sign is reversed under the exchange of two simple roots of $su(3)$, *i.e.* the charge-conjugation ω_c. The representation (4) therefore corresponds to $\omega_c = -1$,

$$\omega_c : \ (0) \mapsto (0) \,, \quad (4) \mapsto -(4) \,. \tag{6.6}$$

From this action, one can express the Ishibashi states of $su(3)_1$ in terms of those of $su(2)_4$ as follows,

$$|(0,0)\rangle\!\rangle = \frac{1}{\sqrt{2}}(|0\rangle\!\rangle + |4\rangle\!\rangle) \,, \quad |(0,0);\omega_c\rangle\!\rangle = \frac{1}{\sqrt{2}}(|0\rangle\!\rangle - |4\rangle\!\rangle) \,, \tag{6.7}$$

where we have used the definition (2.32) of twisted Ishibashi states. Conversely, one obtains

$$\begin{aligned} |0\rangle\!\rangle &= \frac{1}{\sqrt{2}}(|(0,0)\rangle\!\rangle + |(0,0);\omega_c\rangle\!\rangle) = |((0,0);+)\rangle\!\rangle \,, \\ |4\rangle\!\rangle &= \frac{1}{\sqrt{2}}(|(0,0)\rangle\!\rangle - |(0,0);\omega_c\rangle\!\rangle) = |((0,0);-)\rangle\!\rangle \,, \end{aligned} \tag{6.8}$$

where $\pm$ stands for the irreducible representations of $G_c = \{1, \omega_c\} \cong \mathbb{Z}_2$. The remaining two representations of $su(3)_1$ have the trivial stabilizer $\mathcal{S}((1,0)) = \mathcal{S}((0,1)) = \{1\}$ and we have only one basis for each of them

$$|2_1\rangle\!\rangle = |(1,0)\rangle\!\rangle \,, \quad |2_2\rangle\!\rangle = |(0,1)\rangle\!\rangle \,. \tag{6.9}$$

We have distinguished two bulk fields with representation (2) in (6.3) by putting the subscript.

In this way, the basis (4.1) of twisted Ishibashi states in $su(3)_1$ can be regarded as the (untwisted) Ishibashi states of $su(2)_4$. This implies that the regular states (4.5) of the generalized fusion algebra $\mathcal{F}(su(3)_1; G_c)$ yield a complete set of boundary states in (6.3), whose coefficient matrix Ψ is given by the generalized S-matrix (3.11)

$$\Psi = \hat{S} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & \sqrt{2} & \sqrt{2} \\ 1 & 1 & \sqrt{2}\kappa & \sqrt{2}\bar{\kappa} \\ 1 & 1 & \sqrt{2}\bar{\kappa} & \sqrt{2}\kappa \\ \sqrt{3} & -\sqrt{3} & 0 & 0 \end{pmatrix} \quad (\kappa = e^{\frac{2\pi i}{3}})\,. \tag{6.10}$$

Here the columns are ordered as $\{((0,0);+), ((0,0);-), (1,0), (0,1)\}$. The first three rows correspond to three representations $\{(0,0), (1,0), (0,1)\}$ of $su(3)_1$, respectively, while the fourth row stands for the representation of the twisted chiral algebra $A_2^{(2)}$ at level 1. The boundary states of (6.3) that preserve $su(2)_4$ are therefore labeled by $L \in \hat{\mathcal{I}}$ for the generalized fusion algebra $\mathcal{F}(su(3)_1; G_c)$.

The structure constants (6.2) of the graph fusion algebra associated with Ψ is defined by summing over the column indices of Ψ. Since $\Psi = \hat{S}$ in the present case, this is nothing but the generalized Verlinde formula (3.14) if we choose $(0,0) \in \hat{\mathcal{I}}$ as the node 0 in (6.2). Namely, the structure constants of the graph fusion algebra are given by the generalized fusion coefficients of $\mathcal{F}(su(3)_1; G_c)$,

$$G_{LM}{}^{N} = \widehat{\mathcal{N}}_{LM}{}^{N}\,. \tag{6.11}$$

The graph fusion algebra associated with (6.3) therefore coincides with the generalized fusion algebra $\mathcal{F}(su(3)_1; G_c)$. Since the generalized fusion algebra contains the ordinary fusion algebra as a subalgebra, this identification of the graph fusion algebra with the generalized fusion algebra naturally explains the observation that the graph fusion algebra contains the fusion algebra of the extended theory as a subalgebra.

This argument for the relation of graph fusion algebras with the generalized fusion algebras of the extended theory is readily extended to simple current extensions other than $su(2)_4 \subset su(3)_1$. For instance, the case of $su(3)_3 \subset so(8)_1$ in (5.30) can be treated in completely the same way.

First, we give the branching rule for the representations (2.63) of $so(8)_1$,

$$O = (0,0) \oplus (3,0) \oplus (0,3)\,, \quad V = (1,0)\,, \quad S = (1,0)\,, \quad C = (1,0)\,. \tag{6.12}$$

As is shown in Appendix A, the unextended chiral algebra $su(3)_3$ is characterized as the fixed point algebra of an automorphism group $\langle \pi' \rangle \cong \mathbb{Z}_3$ of $so(8)_1$. Since π' has order 3, the eigenvalue of π' takes the form κ^n $(\kappa = e^{\frac{2\pi i}{3}}\,; n = 0,1,2)$. The decomposition (6.12) of the vacuum representation O of $so(8)_1$ into those of $su(3)_3$ corresponds to the decomposition according to the eigenvalue of π', since π' fixes $su(3)_3$. From the explicit realization of $su(3)_3 \subset so(8)_1$ given in Appendix A, one can show that π' acts on the vacuum representation O of $so(8)_1$ as follows

$$\pi' : (0,0) \mapsto (0,0)\,,\ (3,0) \mapsto \bar{\kappa}(3,0)\,,\ (0,3) \mapsto \kappa(0,3)\,. \tag{6.13}$$

The twisted Ishibashi states of $so(8)_1$ are then expressed in terms of the (untwisted) Ishibashi states of $su(3)_3$,

$$\begin{aligned} |O\rangle\!\rangle &= \frac{1}{\sqrt{3}}(|(0,0)\rangle\!\rangle + |(3,0)\rangle\!\rangle + |(0,3)\rangle\!\rangle)\,, \\ |O;\pi'\rangle\!\rangle &= \frac{1}{\sqrt{3}}(|(0,0)\rangle\!\rangle + \bar{\kappa}|(3,0)\rangle\!\rangle + \kappa|(0,3)\rangle\!\rangle)\,, \\ |O;\pi'^2\rangle\!\rangle &= \frac{1}{\sqrt{3}}(|(0,0)\rangle\!\rangle + \kappa|(3,0)\rangle\!\rangle + \bar{\kappa}|(0,3)\rangle\!\rangle)\,, \end{aligned} \tag{6.14}$$

where we have used again the definition (2.32) of twisted Ishibashi states. From this expression, one obtains

$$\begin{aligned} |(0,0)\rangle\!\rangle &= \frac{1}{\sqrt{3}}(|O\rangle\!\rangle + |O;\pi'\rangle\!\rangle + |O;\pi'^2\rangle\!\rangle) = |(O;\rho'_0)\rangle\!\rangle\,, \\ |(3,0)\rangle\!\rangle &= \frac{1}{\sqrt{3}}(|O\rangle\!\rangle + \kappa|O;\pi'\rangle\!\rangle + \bar{\kappa}|O;\pi'^2\rangle\!\rangle) = |(O;\rho'_1)\rangle\!\rangle\,, \\ |(0,3)\rangle\!\rangle &= \frac{1}{\sqrt{3}}(|O\rangle\!\rangle + \bar{\kappa}|O;\pi'\rangle\!\rangle + \kappa|O;\pi'^2\rangle\!\rangle) = |(O;\rho'_2)\rangle\!\rangle\,, \end{aligned} \tag{6.15}$$

where ρ'_n $(n=0,1,2)$ are the irreducible representations of $\langle\pi'\rangle$ defined as $\rho'_n(\pi') = \kappa^{2n}$. The other three representations of $so(8)_1$ have the trivial stabilizer $\{1\} \subset \langle\pi'\rangle$ and we can relate the corresponding Ishibashi states with those of $su(3)_3$ as follows,

$$|(1,1)_1\rangle\!\rangle = |V\rangle\!\rangle\,, \quad |(1,1)_2\rangle\!\rangle = |S\rangle\!\rangle\,, \quad |(1,1)_3\rangle\!\rangle = |C\rangle\!\rangle\,. \tag{6.16}$$

We have distinguished three bulk fields with representation $(1,1)$ in (5.30) again by putting the subscript. The six untwisted Ishibashi states of $su(3)_3$ are thus identified with the six twisted Ishibashi states of $so(8)_1$ for the automorphism group $\langle\pi'\rangle \cong \mathbb{Z}_3$. Hence the boundary states of $su(3)_3$ associated with the modular invariant (5.30) are identified with the twisted boundary states of $so(8)_1$ for $\langle\pi'\rangle \cong \mathbb{Z}_3$.

Since π' is composed of the triality automorphism π and an inner automorphism of $so(8)_1$, the twisted chiral algebra for π' is also isomorphic to the twisted affine Lie algebra $D_4^{(3)}$ at level 1 and we have only one representation for π'. The same argument holds for π'^2. The generalized fusion algebra $\mathcal{F}(so(8)_1;\langle\pi'\rangle)$ therefore consists of $4+1+1=6$ representations. From the definition (3.11), one obtains its generalized S-matrix in the form

$$\hat{S} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 1 & 1 & 1 & \sqrt{3} & -\sqrt{3} & -\sqrt{3} \\ 1 & 1 & 1 & -\sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 1 & 1 & 1 & -\sqrt{3} & -\sqrt{3} & \sqrt{3} \\ 2 & 2\kappa & 2\bar{\kappa} & 0 & 0 & 0 \\ 2 & 2\bar{\kappa} & 2\kappa & 0 & 0 & 0 \end{pmatrix}. \tag{6.17}$$

Here the columns are ordered as $\{(O;\rho'_0),(O;\rho'_1),(O;\rho'_2),(V),(S),(C)\}$. The first four rows express respectively the untwisted representations O,V,S and C, while the last two rows correspond to representations in $\mathcal{I}^{\pi'^2}$ and $\mathcal{I}^{\pi'}$. This matrix gives the boundary state coefficients

for the regular states of $\mathcal{F}(so(8)_1; \langle\pi'\rangle)$. Using the correspondence of Ishibashi states given in (6.15) and (6.16), we can regard this as the boundary state coefficient matrix $\Psi_{D^{(6)}}$ for the untwisted boundary states associated with (5.30). The graph fusion algebra corresponding to $\Psi_{D^{(6)}}$ is therefore identified with the generalized fusion algebra $\mathcal{F}(so(8)_1; \langle\pi'\rangle)$. Since $\mathcal{F}(so(8)_1; \langle\pi'\rangle)$ contains the ordinary fusion algebra $\mathcal{F}(so(8)_1)$, which is the group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2$, the graph fusion algebra of $\Psi_{D^{(6)}}$ also contains it as a subalgebra.

In the modular invariant (5.30), in addition to the untwisted boundary states, we have the twisted boundary states for the charge-conjugation automorphism group $G_c = \{1, \omega_c\}$ of $su(3)_3$. As is shown in Appendix A, the automorphism group G_c has a lift $\tilde{G}_c = \langle\pi', \sigma'\rangle$ to $so(8)_1$, which is isomorphic to the symmetric group S_3. Using this fact, we can construct the ω_c-twisted states in (5.30) starting from the $\tilde{G}_c$-twisted states of $so(8)_1$.

Since $\sigma' \in \tilde{G}_c$ is composed of the chirality flip and an inner automorphism of $so(8)_1$, the twisted chiral algebra for σ' is isomorphic to $D_4^{(2)}$ at level 1 and the generalized fusion algebra $\mathcal{F}(so(8)_1; \tilde{G}_c)$ has exactly the same form as $\mathcal{F}(so(8)_1; S_3)$ presented in Section 2.3. Consequently, one obtains 12 twisted boundary states for $\tilde{G}_c$, of which six states for $\langle\pi'\rangle = \{1, \pi', \pi'^2\}$ correspond to the untwisted states of $su(3)_3$ while the remaining six for the coset $\sigma'\langle\pi'\rangle = \{\sigma', \pi'\sigma', \pi'^2\sigma'\}$ are regarded as the ω_c-twisted ones, since the restriction of $\sigma'\langle\pi'\rangle$ to $su(3)_3$ yields ω_c. For example, two σ'-twisted states of $so(8)_1$ take the form

$$\frac{1}{\sqrt{2}}(|O;\sigma'\rangle\rangle \pm |V;\sigma'\rangle\rangle)\,. \tag{6.18}$$

The σ'-twisted Ishibashi states of $so(8)_1$ can be expressed in terms of the ω_c-twisted Ishibashi states of $su(3)_3$,

$$\begin{aligned} |O;\sigma'\rangle\rangle = R(\sigma')|O\rangle\rangle &= \frac{1}{\sqrt{3}} R(\omega_c)(|(0,0)\rangle\rangle + |(3,0)\rangle\rangle + |(0,3)\rangle\rangle) \\ &= \frac{1}{\sqrt{3}}(|(0,0);\omega_c\rangle\rangle + |(0,3);\omega_c\rangle\rangle + |(3,0);\omega_c\rangle\rangle)\,, \end{aligned} \tag{6.19}$$

$$|V;\sigma'\rangle\rangle = R(\sigma')|V\rangle\rangle = R(\omega_c)|(1,1)_1\rangle\rangle = |(1,1)_1;\omega_c\rangle\rangle\,. \tag{6.20}$$

Substituting these expressions into (6.18), we obtain two ω_c-twisted states of $su(3)_3$,

$$\frac{1}{\sqrt{6}}(|(0,0);\omega_c\rangle\rangle + |(3,0);\omega_c\rangle\rangle + |(0,3);\omega_c\rangle\rangle \pm \sqrt{3}|(1,1)_1;\omega_c\rangle\rangle)\,. \tag{6.21}$$

The case of $\pi'\sigma'$ and $\pi'^2\sigma'$ can be treated in the same way and we obtain four more ω_c-twisted states,

$$\begin{aligned} &\frac{1}{\sqrt{2}}(|O;\pi'\sigma'\rangle\rangle \pm |C;\pi'\sigma'\rangle\rangle) \\ &\quad = \frac{1}{\sqrt{6}}(|(0,0);\omega_c\rangle\rangle + \bar{\kappa}|(3,0);\omega_c\rangle\rangle + \kappa|(0,3);\omega_c\rangle\rangle \pm \sqrt{3}|(1,1)_3;\omega_c\rangle\rangle)\,, \end{aligned} \tag{6.22}$$

$$\begin{aligned} &\frac{1}{\sqrt{2}}(|O;\pi'^2\sigma'\rangle\rangle \pm |S;\pi'^2\sigma'\rangle\rangle) \\ &\quad = \frac{1}{\sqrt{6}}(|(0,0);\omega_c\rangle\rangle + \kappa|(3,0);\omega_c\rangle\rangle + \bar{\kappa}|(0,3);\omega_c\rangle\rangle \pm \sqrt{3}|(1,1)_2;\omega_c\rangle\rangle)\,. \end{aligned} \tag{6.23}$$

The ω_c-twisted boundary states obtained in this way form a complete set of twisted states, since the number of the twisted boundary states is the same as the number $|\mathcal{E}(\omega_c)|$ of the twisted Ishibashi states available in (5.30). The resulting boundary states have the following coefficient matrix

$$\Psi_{D^{(6)*}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & \sqrt{3} & 0 & 0 \\ 1 & 1 & 1 & -\sqrt{3} & 0 & 0 \\ 1 & \kappa & \bar{\kappa} & 0 & \sqrt{3} & 0 \\ 1 & \kappa & \bar{\kappa} & 0 & -\sqrt{3} & 0 \\ 1 & \bar{\kappa} & \kappa & 0 & 0 & \sqrt{3} \\ 1 & \bar{\kappa} & \kappa & 0 & 0 & -\sqrt{3} \end{pmatrix} , \tag{6.24}$$

where the columns are ordered as $\{(0,0),(3,0),(0,3),(1,1)_1,(1,1)_2,(1,1)_3\}$. This matrix $\Psi_{D^{(6)*}}$ realizes the $D^{(6)*}$ NIM-rep of the ordinary fusion algebra $\mathcal{F}(su(3)_3)$. By appropriately mixing three columns of $\Psi_{D^{(6)*}}$ corresponding to $(1,1)_n$ $(n=1,2,3)$, one can make the first row of $\Psi_{D^{(6)*}}$ consist of only non-vanishing entries. The graph fusion algebra for $D^{(6)*}$ is then constructed by taking the first row as the node 0. It would be interesting to relate this algebra with some generalized fusion algebra, possibly $\mathcal{F}(so(8)_1;\tilde{G}_c)$.

7 Summary and Discussion

In this paper, we have studied a consistency condition of boundary states satisfying the boundary condition twisted by an automorphism group G of the chiral algebra $\mathcal{A}$ and clarified the relation of twisted boundary states with the generalized fusion algebra. We have shown that the fusion coefficients of the generalized fusion algebra is expressed by a formula analogous to the Verlinde formula even for non-abelian cases and thereby determined irreducible representations of the generalized fusion algebra, which generalize quantum dimensions of the ordinary fusion algebras. For a non-abelian G, some irreducible representations have dimension greater than 1 reflecting the fact that the generalized fusion algebra is non-commutative. Based on these results, we have shown that a consistent set of twisted boundary states forms a NIM-rep of the generalized fusion algebra. As a check of our argument, we have considered twisted boundary states in several models, which includes non-diagonal modular invariants as well as the case of non-abelian automorphisms. We have seen that several NIM-reps of the ordinary fusion algebra are organized into a single NIM-rep of the generalized fusion algebra. In particular, the D and E type NIM-reps of $su(3)_k$ $(k=3,5)$ are paired with their counterpart D^* and E^*, respectively, to yield a NIM-rep of the generalized fusion algebra for the charge-conjugation automorphism group of $su(3)_k$. Finally, we have given an argument that the graph fusion algebra associated with a simple current extension can be regarded as the generalized fusion algebra of the extended chiral algebra, which naturally explains the observation that the graph fusion algebra contains the fusion algebra of the extended theory as a subalgebra.

Having obtained these results, a natural problem is the classification of NIM-reps of the generalized fusion algebra, which generalizes the corresponding problem [2, 7] for the case of ordinary fusion algebras. These two problems are related with each other, since

the generalized fusion algebra contains the ordinary fusion algebra as a subalgebra and a NIM-rep of the former is decomposed into NIM-reps of the latter. More generally, if the automorphism group G of the chiral algebra $\mathcal{A}$ has a subgroup $H \subsetneq G$, the generalized fusion algebra $\mathcal{F}(\mathcal{A}; G)$ has a subalgebra $\mathcal{F}(\mathcal{A}; H)$, and a NIM-rep of $\mathcal{F}(\mathcal{A}; G)$ is decomposed into NIM-reps of $\mathcal{F}(\mathcal{A}; H)$. On the other hand, as we have pointed out in Section 5.1, there exist some unphysical NIM-reps of the ordinary fusion algebra $\mathcal{F}(u(1)_6)$ that are not originated from those of the generalized fusion algebra $\mathcal{F}(u(1)_6; G_c)$. This example shows that not all of the NIM-reps for $\mathcal{F}(\mathcal{A}; H)$ can be obtained from those of $\mathcal{F}(\mathcal{A}; G)$. It is interesting to find other examples with this property and understand its significance. In particular, any relation with the notion of physical NIM-reps [7] would be desirable.

The case of block diagonal modular invariants is of particular interest. In that case, the chiral algebra $\mathcal{A}$ has an extension $\mathcal{A}_{\text{ext}}$ and one can consider a lift of the automorphism group G of $\mathcal{A}$ to $\mathcal{A}_{\text{ext}}$. If a lift G_{ext} of G exists, [11] one can construct G_{ext}-twisted boundary states of the extended theory, which can also be regarded as G-twisted states of the unextended theory. This suggests that there is some relation between two generalized fusion algebras $\mathcal{F}(\mathcal{A}; G)$ and $\mathcal{F}(\mathcal{A}_{\text{ext}}; G_{\text{ext}})$. Indeed, for the case of simple current extensions, NIM-reps of $\mathcal{F}(\mathcal{A}_{\text{ext}}; G_{\text{ext}})$ can be considered as NIM-reps of $\mathcal{F}(\mathcal{A}; G)$, as we have seen in Section 6. Probably, this correspondence of NIM-reps is a common feature of the cases in which the unextended chiral algebra $\mathcal{A}$ is characterized as the fixed point algebra of a certain automorphism group $H_{\text{ext}} \subset G_{\text{ext}}$ of $\mathcal{A}_{\text{ext}}$. This is exactly the setting of the Galois theory for vertex operator algebras (see [28] and references therein) and the mutual relation of generalized fusion algebras in such cases might be described also by the Galois theory. For exceptional invariants, however, the unextended chiral algebra $\mathcal{A}$ can not be obtained as the fixed point algebra for any automorphism group of $\mathcal{A}_{\text{ext}}$ in general, and one would need another tool, such as the operator-algebraic methods [30, 31, 12], for a deep understanding of the mutual relation of generalized fusion algebras associated with an exceptional invariant.

[11] This is actually an assumption. There is in general an obstruction to lifting the automorphism group G of $\mathcal{A}$ to the extended chiral algebra $\mathcal{A}_{\text{ext}}$ [28].

A Algebra embedding $su(3)_3 \subset so(8)_1$

In this appendix, we give an explicit realization of the algebra embedding $su(3)_3 \subset so(8)_1$ and show that the charge-conjugation automorphism group $G_c = \{1, \omega_c\} \cong \mathbb{Z}_2$ of $su(3)_3$ has a lift $\tilde{G}_c$ to $so(8)_1$, which is isomorphic to the symmetric group S_3.

The affine Lie algebra $so(8)_1$ can be expressed by four pairs of complex fermions $\psi_i^\pm$ $(i = 1,2,3,4)$. The Cartan element $\tilde{H}_i$ and the simple root $\tilde{E}^{\alpha_i}$ of $so(8)$ have the form

$$\tilde{H}_i = \psi_i^+ \psi_i^- \,, \quad \tilde{E}^{\alpha_1} = \psi_1^+ \psi_2^- \,, \quad \tilde{E}^{\alpha_2} = \psi_2^+ \psi_3^- \,, \quad \tilde{E}^{\alpha_3} = \psi_3^+ \psi_4^- \,, \quad \tilde{E}^{\alpha_4} = \psi_3^+ \psi_4^+ \,. \tag{A.1}$$

The other roots are written as $\psi_i^\pm \psi_j^\pm$ or $\psi_i^\pm \psi_j^\mp$ with $i \neq j$. The subalgebra $su(3) \subset so(8)$ with embedding index 3 can be realized as follows,

$$\begin{aligned}
H_1 &= \frac{1}{\sqrt{2}}(-\tilde{H}_1 - 2\tilde{H}_2 + \tilde{H}_3) \,, \quad H_2 = \sqrt{\frac{3}{2}}(\tilde{H}_1 + \tilde{H}_3) \,, \\
E^1 &= \psi_1^- \psi_3^+ + \kappa\, \psi_4^- \psi_2^- + \bar{\kappa}\, \psi_4^+ \psi_2^- \,, \\
E^2 &= \psi_2^+ \psi_3^+ + \kappa\, \psi_1^+ \psi_4^+ + \bar{\kappa}\, \psi_1^+ \psi_4^- \,, \\
E^\theta &= \tilde{E}^{\alpha_1} + \tilde{E}^{\alpha_3} + \tilde{E}^{\alpha_4} = \psi_1^+ \psi_2^- + \psi_3^+ \psi_4^- + \psi_3^+ \psi_4^+ \,,
\end{aligned} \tag{A.2}$$

where $H_i\,(i = 1,2)$ and $E^i (i = 1,2)$ stand for the Cartan elements and the simple roots of $su(3)$, respectively. $E^\theta = [E^1, E^2]$ is the highest root of $su(3)$ and $\kappa = e^{\frac{2\pi i}{3}}$. The 28 elements of $so(8)$ can be decomposed into irreducible representations of this $su(3)$ as $\mathbf{28} = \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}$, where $\mathbf{10}$ ($\overline{\mathbf{10}}$) is the highest weight representation with the Dynkin label $(3,0)$ ($(0,3)$) and contains $\psi_2^- \psi_3^+$ ($\psi_1^+ \psi_3^+$) as the highest weight state.

One can construct an automorphism of $so(8)$ that fixes the $su(3)$ subalgebra (A.2) from the triality of $so(8)$. Let π be the triality automorphism of $so(8)$, which generates an automorphism group $\langle \pi \rangle = \{1, \pi, \pi^2\} \cong \mathbb{Z}_3$. The action of π on the simple roots of $so(8)$ reads

$$\pi : \tilde{E}^{\alpha_1} \mapsto \tilde{E}^{\alpha_3} \mapsto \tilde{E}^{\alpha_4} \mapsto \tilde{E}^{\alpha_1} \,, \ \tilde{E}^{\alpha_2} \mapsto \tilde{E}^{\alpha_2} \,, \tag{A.3}$$

which in turn implies the following action on the Cartan elements,

$$\begin{aligned}
\pi : \tilde{H}_1 &\mapsto \frac{1}{2}(\tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 - \tilde{H}_4) \,, \\
\tilde{H}_2 &\mapsto \frac{1}{2}(\tilde{H}_1 + \tilde{H}_2 - \tilde{H}_3 + \tilde{H}_4) \,, \\
\tilde{H}_3 &\mapsto \frac{1}{2}(\tilde{H}_1 - \tilde{H}_2 + \tilde{H}_3 + \tilde{H}_4) \,, \\
\tilde{H}_4 &\mapsto \frac{1}{2}(\tilde{H}_1 - \tilde{H}_2 - \tilde{H}_3 - \tilde{H}_4) \,.
\end{aligned} \tag{A.4}$$

The action on the other elements of $so(8)$ can be determined by the commutation relations and one obtains

$$\begin{aligned}
\pi : &\psi_1^+ \psi_3^- \mapsto \psi_4^- \psi_2^+ \mapsto \psi_4^+ \psi_2^+ \mapsto \psi_1^+ \psi_3^- \,, \ \psi_2^+ \psi_3^+ \mapsto \psi_1^+ \psi_4^+ \mapsto \psi_1^+ \psi_4^- \mapsto \psi_2^+ \psi_3^+ \,, \\
&\psi_1^+ \psi_3^+ \mapsto \psi_1^+ \psi_3^+ \,, \ \psi_1^+ \psi_2^+ \mapsto \psi_1^+ \psi_2^+ \,.
\end{aligned} \tag{A.5}$$

Here we give only the action on the positive roots; the action on the negative roots can be obtained by taking the hermitian conjugation of this equation. Since the order of π is 3, its eigenvalue is of the form κ^n $(n = 0, 1, 2)$, and $so(8)$ is decomposed into the eigenspaces of π as $\mathbf{28} = \mathbf{14} \oplus \mathbf{7} \oplus \mathbf{7}$. The fixed point algebra of π has dimension 14 and coincides with $G_2 \subset so(8)$ with embedding index 1.

π acts on $su(3)$ in (A.2) as follows,

$$\pi : E^1 \mapsto \kappa\, E^1\ ,\ E^2 \mapsto \bar{\kappa}\, E^2\ ,\ E^\theta \mapsto E^\theta\ ,\ H_i \mapsto H_i\ . \tag{A.6}$$

Hence π almost fixes $su(3)$. One can construct an automorphism π' fixing $su(3)$ in the form $\pi' = \mathrm{Ad}_h \pi$, where Ad_h is an inner automorphism $J \mapsto \mathrm{Ad}_h(J) = hJh^{-1}$ with the element $h = e^{\frac{2\pi i}{3}(H_1+H_2)} \in SO(8)$. From the action of Ad_h on $\psi_i^\pm$,

$$\mathrm{Ad}_h : \psi_1^\pm \mapsto \kappa^{\pm 1}\psi_1^\pm\ ,\ \psi_2^\pm \mapsto \kappa^{\pm 1}\psi_2^\pm\ ,\ \psi_3^\pm \mapsto \psi_3^\pm\ ,\ \psi_4^\pm \mapsto \psi_4^\pm\ , \tag{A.7}$$

one can readily check that $\pi' = \mathrm{Ad}_h\pi$ actually fixes $su(3)$. Since Ad_h commutes with π and $(\mathrm{Ad}_h)^3 = 1$, the automorphism π' has also order 3 and hence generates an automorphism group $\langle \pi' \rangle = \{1, \pi', \pi'^2\} \cong \mathbb{Z}_3$. The elements of $so(8)$ are again decomposed according to the eigenvalue κ^n $(n = 0, 1, 2)$ of π' as $\mathbf{28} = \mathbf{8} \oplus \overline{\mathbf{10}} \oplus \mathbf{10}$, which coincides with the decomposition into irreducible representations of $su(3)$ mentioned above. In this way, the $su(3)$ subalgebra (A.2) is characterized as the fixed point algebra of the automorphism group $\langle \pi' \rangle \cong \mathbb{Z}_3$ of $so(8)$.

The fact that $su(3)$ in (A.2) is characterized as the fixed point algebra of π' implies that the identity automorphism of $su(3)$ has a lift $\langle \pi' \rangle \cong \mathbb{Z}_3$ to $so(8)$. One can show that the charge-conjugation automorphism ω_c of $su(3)$ also has a lift to $so(8)$. Consider the following automorphism σ' of $so(8)$,

$$\sigma' : \psi_1^\pm \mapsto \psi_2^\mp\ ,\ \psi_2^\pm \mapsto \psi_1^\mp\ ,\ \psi_3^\pm \mapsto \psi_3^\pm\ ,\ \psi_4^\pm \mapsto -\psi_4^\mp\ , \tag{A.8}$$

which has order 2. This is an outer automorphism of $so(8)$, since σ' acts on eight real fermions as an element of $O(8)$ with determinant -1. This automorphism σ' acts on $su(3)$ in (A.2) as the exchange of two simple roots

$$\sigma' : E^1 \mapsto E^2\ ,\ E^\theta \mapsto -E^\theta\ . \tag{A.9}$$

In particular, σ' keeps $su(3) \subset so(8)$ invariant and can be restricted to $su(3)$. Clearly, the restriction of σ' to $su(3)$ is the charge-conjugation ω_c of $su(3)$. In other words, σ' is a lift of ω_c to $so(8)$. Together with π', σ' forms a lift $\tilde{G}_c$ of $G_c = \{1, \omega_c\}$. By explicit calculations, one can show that $\sigma'\pi'\sigma'^{-1} = \pi'^{-1}$, which means that $\tilde{G}_c = \langle \pi', \sigma' \rangle$ is isomorphic to the symmetric group S_3; the charge-conjugation automorphism group G_c of $su(3) \subset so(8)$ has a lift $\tilde{G}_c$ to $so(8)$, which is isomorphic to S_3. The restriction G_c of $\tilde{G}_c$ can be identified with the quotient group of $\tilde{G}_c$ by the stabilizer of $su(3) \subset so(8)$ as $\tilde{G}_c/\langle \pi' \rangle \cong S_3/\mathbb{Z}_3 \cong \mathbb{Z}_2$.

B Twisted boundary states for $su(3)_k$

In this appendix, we describe in detail the construction of twisted boundary states for the chiral algebra $su(3)_k$ $(k = 3, 5)$ and the charge-conjugation automorphism group $G_c = \{1, \omega_c\}$,

and check that the resulting boundary states realize a NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_k; G_c)$.

Let us first calculate the explicit form of the generalized fusion algebra $\mathcal{F}(su(3)_k; G_c)$ for $k = 3$ and 5. The set $\mathcal{I}$ of the irreducible representations of $su(3)_k$ reads

$$\mathcal{I} = P_+^k(su(3)) = \{\lambda = (\lambda_1, \lambda_2) | \lambda_1 + \lambda_2 \leq k\}. \tag{B.1}$$

The modular transformation matrix of the characters of these representations is given by the Kac-Peterson formula [32]. The charge conjugation ω_c acts on $\mathcal{I}$ as

$$\omega_c \ : \ (\lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1). \tag{B.2}$$

The corresponding twisted chiral algebra is the twisted affine Lie algebra $A_2^{(2)}$. Since its horizontal subalgebra is $so(3)$, the irreducible representations of $A_2^{(2)}$ are labeled by a single Dynkin label which we denote by $\tilde{\lambda}$,

$$\mathcal{I}^{\omega_c} = P_+^k(A_2^{(2)}) = \{\tilde{\lambda} \mid 2\tilde{\lambda} \leq k\}. \tag{B.3}$$

The set $\mathcal{I}(\omega_c)$ of representations fixed by ω_c reads

$$\mathcal{I}(\omega_c) = \{\mu = (\mu_1, \mu_1) \mid 2\mu_1 \leq k\}. \tag{B.4}$$

The modular transformation matrix S^{ω_c} of the twisted representation $\tilde{\lambda} \in \mathcal{I}^{\omega_c}$ takes the form [32]

$$S^{\omega_c}_{\tilde{\lambda}\mu} = \frac{2}{\sqrt{k+3}} \sin\Big(\frac{2\pi}{k+3}(\tilde{\lambda}+1)(\mu_1+1)\Big). \tag{B.5}$$

By using these data, one can construct untwisted and ω_c-twisted boundary states compatible with the charge conjugation modular invariant of $su(3)_k$. The total number $|\hat{\mathcal{I}}|$ of boundary states is

$$|\hat{\mathcal{I}}| = |\mathcal{I}| + |\mathcal{I}^{\omega_c}| = \begin{cases} 10 + 2 = 12 & \text{for} \quad k = 3, \\ 21 + 3 = 24 & \text{for} \quad k = 5. \end{cases} \tag{B.6}$$

As we have shown in Section 2, the generalized fusion algebra $\mathcal{F}(su(3)_k; G_c)$, or equivalently the regular NIM-reps of $\mathcal{F}(su(3)_k; G_c)$, is obtained from the overlaps of these boundary states. We give only the part of $\mathcal{F}(su(3)_k; G_c)$ concerning the twisted representations. The part containing only the untwisted representations, *i.e.*, the ordinary fusion algebra $\mathcal{F}(su(3)_k)$, is found in, *e.g.*, [33].

$\underline{k = 3}$

$$\begin{aligned}
(\lambda_1, \lambda_2) \times (0) = (\lambda_1, \lambda_2) \times (1) &= \begin{cases} (0) & (\lambda_1, \lambda_2) \in \{(0,0), (3,0), (0,3)\}, \\ (0) + (1) & (\lambda_1, \lambda_2) \in \{(1,0), (0,1), (2,0), (0,2), (2,1), (1,2)\}, \end{cases} \\
(1,1) \times (0) &= (0) + 2\,(1), \\
(1,1) \times (1) &= 2\,(0) + (1), \\
(0) \times (0) = (1) \times (1) &= \sum_{(\lambda_1,\lambda_2)\in\mathcal{I}} (\lambda_1, \lambda_2), \\
(0) \times (1) &= (1,0) + (0,1) + (2,0) + (0,2) + (2,1) + (1,2) + 2\,(1,1).
\end{aligned} \tag{B.7}$$

$\underline{k=5}$

$$
\begin{aligned}
&(\lambda_1,\lambda_2)\times(\tilde{\lambda})=(\tilde{\lambda}) && (\lambda_1,\lambda_2)\in\{(0,0),(5,0),(0,5)\},\quad \tilde{\lambda}\in\mathcal{I}^{\omega_c}=\{0,1,2\},\\
&(\lambda_1,\lambda_2)\times\begin{cases}(0)=(0)+(1)\\(1)=(0)+(1)+(2)\\(2)=(1)+(2)\end{cases} && (\lambda_1,\lambda_2)\in\{(1,0),(0,1),(4,0),(0,4),(4,1),(1,4)\},\\
&(\lambda_1,\lambda_2)\times\begin{cases}(0)=(0)+(1)+(2)\\(1)=(0)+2\,(1)+(2)\\(2)=(0)+(1)+(2)\end{cases} && (\lambda_1,\lambda_2)\in\{(2,0),(0,2),(3,0),(0,3),(3,2),(2,3)\},\\
&(\lambda_1,\lambda_2)\times\begin{cases}(0)=(0)+2\,(1)+(2)\\(1)=2\,(0)+2\,(1)+2\,(2)\\(2)=(0)+2\,(1)+(2)\end{cases} && (\lambda_1,\lambda_2)\in\{(1,1),(3,1),(1,3)\},\\
&(\lambda_1,\lambda_2)\times\begin{cases}(0)=(0)+2\,(1)+2\,(2)\\(1)=2\,(0)+3\,(1)+2\,(2)\\(2)=2\,(0)+2\,(1)+(2)\end{cases} && (\lambda_1,\lambda_2)\in\{(2,1),(1,2),(2,2)\},\\
&(0)\times(0)=(2)\times(2)=\sum_{(\lambda_1,\lambda_2)\in\mathcal{I}}(\lambda_1,\lambda_2),\\
&(1)\times(1)=(0,0)+(5,0)+(0,5)+(1,0)+(0,1)+(4,0)+(0,4)+(4,1)+(1,4)\\
&\qquad\qquad+2\Big((2,0)+(0,2)+(3,0)+(0,3)+(3,2)+(2,3)+(1,1)+(3,1)+(1,3)\Big)\\
&\qquad\qquad+3\Big((2,1)+(1,2)+(2,2)\Big),\\
&(0)\times(1)=(1)\times(2)=(1,0)+(0,1)+(4,0)+(0,4)+(4,1)+(1,4)\\
&\qquad\qquad\qquad\qquad+(2,0)+(0,2)+(3,0)+(0,3)+(3,2)+(2,3)\\
&\qquad\qquad\qquad\qquad+2\Big((1,1)+(3,1)+(1,3)+(2,1)+(1,2)+(2,2)\Big),\\
&(0)\times(2)=(2,0)+(0,2)+(3,0)+(0,3)+(3,2)+(2,3)+(1,1)+(3,1)+(1,3)\\
&\qquad\qquad+2\Big((2,1)+(1,2)+(2,2)\Big).
\end{aligned}
\tag{B.8}
$$

B.1 $su(3)_3$

The modular invariant

$$
Z_{D^{(6)}}=|\chi_{(0,0)}+\chi_{(3,0)}+\chi_{(0,3)}|^2+3|\chi_{(1,1)}|^2 \tag{B.9}
$$

is a simple current extension of $su(3)_3$ by $(3,0)$ and the corresponding extended chiral algebra is $so(8)_1$. As we have shown in Section 6, the twisted boundary states for the automorphism group G_c are obtained from those for $\tilde{G}_c$, a lift of G_c to $so(8)_1$. The resulting boundary state coefficients are given in (6.17) and (6.24). The set $\hat{\mathcal{V}}$ of all the twisted boundary states consists of 12 elements

$$
|\hat{\mathcal{V}}|=|\mathcal{V}|+|\mathcal{V}^{\omega_c}|=6+6=12. \tag{B.10}
$$

By calculating the overlaps of these 12 boundary states, we obtain a set of 12×12 matrices $\{\hat{n}_N \,|N \in \hat{\mathcal{I}}\}$ defined in (4.12). These matrices turn out to be non-negative integer valued. The explicit form is as follows,

$$\begin{aligned}
&\hat{n}_{(0,0)} = \hat{n}_{(3,0)} = \hat{n}_{(0,3)} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}, \\
&\hat{n}_{(1,0)} = \hat{n}_{(0,2)} = \hat{n}_{(2,1)} = \hat{n}^T_{(0,1)} = \hat{n}^T_{(2,0)} = \hat{n}^T_{(1,2)} = \begin{pmatrix} \varrho & O \\ O & \varrho' \end{pmatrix}, \\
&\hat{n}_{(1,1)} = \begin{pmatrix} \varrho_{(1,1)} & O \\ O & \varrho'_{(1,1)} \end{pmatrix}, \\
&\hat{n}_{(0)} = \begin{pmatrix} O & \varrho_0^T \\ \varrho_0 & O \end{pmatrix}, \\
&\hat{n}_{(1)} = \begin{pmatrix} O & \varrho_1^T \\ \varrho_1 & O \end{pmatrix},
\end{aligned} \tag{B.11}$$

where the first six rows and columns stand for the states of $\Psi_{D^{(6)}}$ in (6.17) and the last six for those of $\Psi_{D^{(6)*}}$ in (6.24). O and I are the 6×6 zero and the unit matrices, respectively, and the other matrices are defined as

$$\begin{aligned}
&\varrho = \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&2 \\ 1&1&1&1&0&0 \end{pmatrix}, \quad
\varrho' = \begin{pmatrix} 0&0&0&0&1&1 \\ 0&0&0&0&1&1 \\ 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&1&1&0&0 \end{pmatrix}, \\
&\varrho_{(1,1)} = \begin{pmatrix} 0&1&1&1&0&0 \\ 1&0&1&1&0&0 \\ 1&1&0&1&0&0 \\ 1&1&1&0&0&0 \\ 0&0&0&0&3&0 \\ 0&0&0&0&0&3 \end{pmatrix}, \quad
\varrho'_{(1,1)} = \begin{pmatrix} 1&2&0&0&0&0 \\ 2&1&0&0&0&0 \\ 0&0&1&2&0&0 \\ 0&0&2&1&0&0 \\ 0&0&0&0&1&2 \\ 0&0&0&0&2&1 \end{pmatrix}, \\
&\varrho_0 = \begin{pmatrix} 1&1&0&0&1&1 \\ 0&0&1&1&1&1 \\ 1&0&1&0&1&1 \\ 0&1&0&1&1&1 \\ 1&0&0&1&1&1 \\ 0&1&1&0&1&1 \end{pmatrix}, \quad
\varrho_1 = \begin{pmatrix} 0&0&1&1&1&1 \\ 1&1&0&0&1&1 \\ 0&1&0&1&1&1 \\ 1&0&1&0&1&1 \\ 0&1&1&0&1&1 \\ 1&0&0&1&1&1 \end{pmatrix}.
\end{aligned} \tag{B.12}$$

One can check that these matrices $\hat{n}$ form a NIM-rep of $\mathcal{F}(su(3)_3; G_c)$ defined in (B.7). For example, $\hat{n}$ satisfies the following relation

$$\begin{aligned}
&\hat{n}_{(1,1)}\,\hat{n}_{(0)} = \hat{n}_{(0)} + 2\,\hat{n}_{(1)}, \quad \hat{n}_{(1,1)}\,\hat{n}_{(1)} = 2\,\hat{n}_{(0)} + \hat{n}_{(1)}, \\
&\hat{n}_{(0)}\,\hat{n}_{(1)} = \hat{n}_{(1,0)} + \hat{n}_{(0,1)} + \hat{n}_{(2,0)} + \hat{n}_{(0,2)} + \hat{n}_{(2,1)} + \hat{n}_{(1,2)} + 2\,\hat{n}_{(1,1)}, \quad \text{etc.}
\end{aligned} \tag{B.13}$$

In this way, two NIM-reps, $D^{(6)}$ and $D^{(6)*}$, associated with the simple current modular invariant (B.9) are combined to yield a NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_3; G_c)$.

B.2 $su(3)_5$

We next turn to the construction of twisted boundary states associated with a non-trivial modular invariant of $su(3)_5$ and check that the resulting boundary states realize a NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_5; G_c)$.

B.2.1 The case of $Z_{E^{(8)}}$

We first consider the exceptional modular invariant of $su(3)_5$,

$$\begin{aligned} Z_{E^{(8)}} &= |\chi_{(0,0)} + \chi_{(2,2)}|^2 + |\chi_{(0,2)} + \chi_{(3,2)}|^2 + |\chi_{(1,2)} + \chi_{(5,0)}|^2 \\ &\quad + |\chi_{(3,0)} + \chi_{(0,3)}|^2 + |\chi_{(2,1)} + \chi_{(0,5)}|^2 + |\chi_{(2,0)} + \chi_{(2,3)}|^2 . \end{aligned} \tag{B.14}$$

This invariant is originated from the charge conjugation modular invariant of $su(6)_1$ through the conformal embedding $su(3)_5 \subset su(6)_1$, in which the representations of $su(6)_1$ branch to those of $su(3)_5$ as follows,

$$\begin{aligned} \Lambda_0 &\mapsto (0,0) \oplus (2,2), \\ \Lambda_1 &\mapsto (0,2) \oplus (3,2), \\ \Lambda_2 &\mapsto (1,2) \oplus (5,0), \\ \Lambda_3 &\mapsto (3,0) \oplus (0,3), \\ \Lambda_4 &\mapsto (2,1) \oplus (0,5), \\ \Lambda_5 &\mapsto (2,0) \oplus (2,3). \end{aligned} \tag{B.15}$$

In the modular invariant (B.14), there are twelve untwisted Ishibashi states available, [12]

$$\mathcal{E} = \{(0,0),(5,0),(0,5),(2,2),(1,2),(2,1),(3,0),(2,3),(0,2),(0,3),(2,0),(3,2)\}. \tag{B.16}$$

Correspondingly, we have twelve untwisted boundary states. Six of them are identified with the untwisted states of $su(6)_1$ and the other six states are generated by using the fusion with the representations of $su(3)_5$. The resulting boundary state coefficient matrix $\Psi_{E^{(8)}}$ takes the following form

$$\Psi_{E^{(8)}} = \frac{1}{2}\begin{pmatrix} a_1 K & a_2\, K & K & K \\ a_1 K & a_2\, K & -\,K & -\,K \\ a_2 K & -a_1\, K & i\, K & -i\, K \\ a_2 K & -a_1\, K & -i\, K & i\, K \end{pmatrix}, \tag{B.17}$$

where $a_1 = \sqrt{2}\sin\frac{\pi}{8} = \sqrt{\frac{2-\sqrt{2}}{2}}$, $a_2 = \sqrt{2}\cos\frac{\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{2}}$ and the matrix K is defined as

$$K = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{pmatrix}. \tag{B.18}$$

[12] For simplicity, we omit in this appendix the label for the anti-holomorphic representation of Ishibashi states.

Here the columns of $\Psi_{E^{(8)}}$ are ordered as in (B.16). The first six rows correspond to the states of the $su(6)_1$ theory. One can check that these untwisted states yield the $E^{(8)}$ NIM-rep of the ordinary fusion algebra $\mathcal{F}(su(3)_5)$.

The labels of the available ω_c-twisted Ishibashi states are obtained from (B.14),

$$\mathcal{E}(\omega_c) = \{(0,0),(2,2),(0,3),(3,0)\}. \tag{B.19}$$

As is seen from the branching rule (B.15), the charge conjugation ω_c can be lifted to the charge conjugation $\tilde{\omega_c}$ of $su(6)_1$, which acts on the fundamental weights of $su(6)_1$ as follows,

$$\tilde{\omega_c} \;:\; \Lambda_0 \mapsto \Lambda_0\,, \quad \Lambda_i \leftrightarrow \Lambda_{6-i} \quad (i=1,2,3,4,5). \tag{B.20}$$

The $\tilde{\omega}_c$-twisted boundary states in the $su(6)_1$ theory take the form

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|\Lambda_0;\tilde{\omega}_c\rangle\!\rangle \pm |\Lambda_3;\tilde{\omega}_c\rangle\!\rangle). \tag{B.21}$$

One can express these states in terms of the ω-twisted Ishibashi states of $su(3)_5$ in the same way as the $D^{(6)*}$ states considered in Section 6. First, from the branching rule (B.15) and the modular transformation matrix of $su(3)_5$, the Ishibashi state for the vacuum representation of $su(6)_1$ can be decomposed as follows,

$$|\Lambda_0\rangle\!\rangle = \frac{1}{\sqrt{2}}(a_1|(0,0)\rangle\!\rangle + a_2|(2,2)\rangle\!\rangle)\,, \tag{B.22}$$

where a_1, a_2 are the constants used in (B.17). The $\tilde{\omega}_c$-twisted Ishibashi state $|\Lambda_0;\tilde{\omega}_c\rangle\!\rangle$ is then obtained by the action of $\tilde{\omega}_c$,

$$|\Lambda_0;\tilde{\omega}_c\rangle\!\rangle = R(\tilde{\omega}_c)|\Lambda_0\rangle\!\rangle = \frac{1}{\sqrt{2}}(a_1|(0,0);\omega_c\rangle\!\rangle - a_2|(2,2);\omega_c\rangle\!\rangle)\,. \tag{B.23}$$

Here we use the fact that the $\tilde{\omega}_c$ acts on $(2,2)$ as -1, which can be shown, for example, by the calculation of $\langle\Lambda_0|\tilde{q}^{\frac{1}{2}H_c}|+\rangle$. In this way, we obtain two ω_c-twisted states. The remaining two states are generated by the fusion of $su(3)_5$ and we obtain four ω_c-twisted states, whose boundary state coefficient matrix takes the form

$$\Psi_{E^{(8)*}} = \frac{1}{2}\begin{pmatrix} a_1 & -a_2 & 1 & 1 \\ a_1 & -a_2 & -1 & -1 \\ a_2 & a_1 & i & -i \\ a_2 & a_1 & -i & i \end{pmatrix}\,. \tag{B.24}$$

Here the columns are ordered as in (B.19) and the first (second) row corresponds to $|+\rangle$ ($|-\rangle$). These states form the $E^{(8)*}$ NIM-rep of $\mathcal{F}(su(3)_5)$.

The number of all the twisted boundary states is thus

$$|\hat{\mathcal{V}}| = |\mathcal{V}| + |\mathcal{V}^{\omega_c}| = 12 + 4 = 16\,. \tag{B.25}$$

From the explicit form of the boundary state coefficients, one can calculate the corresponding overlap matrices $\hat{n}$, which are 16-dimensional. The result is as follows,

$$
\begin{aligned}
&\hat{n}_{(0,0)} = I_{16\times 16},\\
&\hat{n}_{(1,0)} = \left(\begin{array}{cccc|cc}
O & O & \varsigma & O & & \\
O & O & O & \varsigma & \multicolumn{2}{c}{\mathbf{O}} \\
O & \varsigma & \varsigma & \varsigma & & \\
\varsigma & O & \varsigma & \varsigma & & \\ \hline
 & & & & O' & I' \\
\multicolumn{4}{c|}{\mathbf{O}} & \varsigma' & I'+\varsigma'
\end{array}\right),\\
&\hat{n}_{(2,0)} = \left(\begin{array}{cccc|cc}
O & \varsigma^t & \varsigma^t & O & & \\
\varsigma^t & O & O & \varsigma^t & \multicolumn{2}{c}{\mathbf{O}} \\
O & \varsigma^t & \varsigma^t & 2\,\varsigma^t & & \\
\varsigma^t & O & 2\,\varsigma^t & \varsigma^t & & \\ \hline
 & & & & \varsigma' & I' \\
\multicolumn{4}{c|}{\mathbf{O}} & \varsigma' & I'+2\,\varsigma'
\end{array}\right), \hat{n}_{(1,1)} = \left(\begin{array}{cccc|cc}
O & O & I & I & & \\
O & O & I & I & \multicolumn{2}{c}{\mathbf{O}} \\
I & I & 2I & 2I & & \\
I & I & 2I & 2I & & \\ \hline
 & & & & O' & I'+\varsigma' \\
\multicolumn{4}{c|}{\mathbf{O}} & I'+\varsigma' & 2(I'+\varsigma')
\end{array}\right),\\
&\hat{n}_{(3,0)} = \left(\begin{array}{cccc|cc}
O & I & O & I & & \\
I & O & I & O & \multicolumn{2}{c}{\mathbf{O}} \\
I & O & I & 2I & & \\
O & I & 2I & I & & \\ \hline
 & & & & \varsigma' & \varsigma' \\
\multicolumn{4}{c|}{\mathbf{O}} & I' & I'+2\,\varsigma'
\end{array}\right), \hat{n}_{(2,1)} = \left(\begin{array}{cccc|cc}
\varsigma & O & \varsigma & \varsigma & & \\
O & \varsigma & \varsigma & \varsigma & \multicolumn{2}{c}{\mathbf{O}} \\
\varsigma & \varsigma & 3\,\varsigma & 2\,\varsigma & & \\
\varsigma & \varsigma & 2\,\varsigma & 3\,\varsigma & & \\ \hline
 & & & & I' & I'+\varsigma' \\
\multicolumn{4}{c|}{\mathbf{O}} & I'+\varsigma' & 3I'+2\,\varsigma'
\end{array}\right),\\
&\hat{n}_{(4,0)} = \left(\begin{array}{cccc|cc}
O & O & O & \varsigma & & \\
O & O & \varsigma & O & \multicolumn{2}{c}{\mathbf{O}} \\
\varsigma & O & \varsigma & \varsigma & & \\
O & \varsigma & \varsigma & \varsigma & & \\ \hline
 & & & & O' & \varsigma' \\
\multicolumn{4}{c|}{\mathbf{O}} & I' & I'+\varsigma'
\end{array}\right), \hat{n}_{(3,1)} = \left(\begin{array}{cccc|cc}
O & O & \varsigma^t & \varsigma^t & & \\
O & O & \varsigma^t & \varsigma^t & \multicolumn{2}{c}{\mathbf{O}} \\
\varsigma^t & \varsigma^t & 2\,\varsigma^t & 2\,\varsigma^t & & \\
\varsigma^t & \varsigma^t & 2\,\varsigma^t & 2\,\varsigma^t & & \\ \hline
 & & & & O' & I'+\varsigma' \\
\multicolumn{4}{c|}{\mathbf{O}} & I'+\varsigma' & 2(I'+\varsigma')
\end{array}\right),\\
&\hat{n}_{(2,2)} = \left(\begin{array}{cccc|cc}
I & O & I & I & & \\
O & I & I & I & \multicolumn{2}{c}{\mathbf{O}} \\
I & I & 3I & 2I & & \\
I & I & 2I & 3I & & \\ \hline
 & & & & I' & I'+\varsigma' \\
\multicolumn{4}{c|}{\mathbf{O}} & I'+\varsigma' & 3I'+2\,\varsigma'
\end{array}\right), \hat{n}_{(5,0)} = \left(\begin{array}{cccc|cc}
\varsigma^t & O & O & O & & \\
O & \varsigma^t & O & O & \multicolumn{2}{c}{\mathbf{O}} \\
O & O & \varsigma^t & O & & \\
O & O & O & \varsigma^t & & \\ \hline
 & & & & I' & O' \\
\multicolumn{4}{c|}{\mathbf{O}} & O' & I'
\end{array}\right),\\
&\hat{n}_{(4,1)} = \left(\begin{array}{cccc|cc}
O & O & I & O & & \\
O & O & O & I & \multicolumn{2}{c}{\mathbf{O}} \\
O & I & I & I & & \\
I & O & I & I & & \\ \hline
 & & & & O' & I' \\
\multicolumn{4}{c|}{\mathbf{O}} & \varsigma' & I'+\varsigma'
\end{array}\right), \hat{n}_{(3,2)} = \left(\begin{array}{cccc|cc}
O & \varsigma & \varsigma & O & & \\
\varsigma & O & O & \varsigma & \multicolumn{2}{c}{\mathbf{O}} \\
O & \varsigma & \varsigma & 2\,\varsigma & & \\
\varsigma & O & 2\,\varsigma & \varsigma & & \\ \hline
 & & & & \varsigma' & I' \\
\multicolumn{4}{c|}{\mathbf{O}} & \varsigma' & I'+2\,\varsigma'
\end{array}\right),\\
&\hat{n}_{(0)} = \hat{n}_{(2)} = \begin{pmatrix} \mathbf{O} & A^t \\ A & \mathbf{O} \end{pmatrix}, \; \hat{n}_{(1)} = \begin{pmatrix} \mathbf{O} & B^t \\ B & \mathbf{O} \end{pmatrix}.
\end{aligned}
\tag{B.26}
$$

The matrices not presented above are obtained by $\hat{n}_{(\lambda_2,\lambda_1)} = \hat{n}^T_{(\lambda_1,\lambda_2)}$. The first 12 rows and columns of these matrices $\hat{n}$ are ordered as in the rows of $\Psi_{E^{(8)}}$ in (B.17) and the last four are ordered as in $\Psi_{E^{(8)*}}$ defined in (B.24). O and O' (I and I') are respectively 3×3 and 2×2 zero (unit) matrices while the other matrices are defined as

$$\varsigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; \varsigma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 0_6 & 1_6 \\ 0_6 & 1_6 \\ 1_6 & 2_6 \\ 1_6 & 2_6 \end{pmatrix}, \; B = \begin{pmatrix} 1_6 & 1_6 \\ 1_6 & 1_6 \\ 1_6 & 3_6 \\ 1_6 & 3_6 \end{pmatrix}, \tag{B.27}$$

where $m_6\,(m = 0,1)$ is a row vector with all entries being m, $m_6 = (m,m,m,m,m,m)$. One can explicitly check that these non-negative integer valued matrices $\{\hat{n}_N\}$ satisfy the generalized fusion algebra $\mathcal{F}(su(3)_5; G_c)$ in (B.8), for example,

$$\begin{aligned}
&\hat{n}_{(2,2)}\,\hat{n}_{(0)} = \hat{n}_{(0)} + 2\,\hat{n}_{(1)} + 2\,\hat{n}_{(2)}, \\
&\hat{n}_{(2,2)}\,\hat{n}_{(1)} = 2\,\hat{n}_{(0)} + 3\,\hat{n}_{(1)} + 2\,\hat{n}_{(2)}, \\
&\hat{n}_{(2,2)}\,\hat{n}_{(2)} = 2\,\hat{n}_{(0)} + 2\,\hat{n}_{(1)} + \hat{n}_{(2)}, \\
&\hat{n}_{(0)}\,\hat{n}_{(2)} = \hat{n}_{(2,0)} + \hat{n}_{(0,2)} + \hat{n}_{(3,0)} + \hat{n}_{(0,3)} + \hat{n}_{(3,2)} + \hat{n}_{(2,3)} + \hat{n}_{(1,1)} + \hat{n}_{(3,1)} + \hat{n}_{(1,3)} \\
&\qquad\qquad + 2\big(\hat{n}_{(2,1)} + \hat{n}_{(1,2)} + \hat{n}_{(2,2)}\big) \qquad \text{etc.}
\end{aligned} \tag{B.28}$$

B.2.2 The case of $Z_{D^{(8)}}$

We next consider twisted boundary states associated with the simple current invariant

$$\begin{aligned}
Z_{D^{(8)}} =& |\chi_{(0,0)}|^2 + |\chi_{(1,1)}|^2 + |\chi_{(2,2)}|^2 + |\chi_{(3,0)}|^2 + |\chi_{(0,3)}|^2 + |\chi_{(4,1)}|^2 + |\chi_{(1,4)}|^2 \\
&+ \big(\chi_{(5,0)}\overline{\chi_{(0,5)}} + \chi_{(3,1)}\overline{\chi_{(1,3)}} + \chi_{(1,2)}\overline{\chi_{(2,1)}} \\
&+ \chi_{(2,3)}\overline{\chi_{(0,2)}} + \chi_{(2,0)}\overline{\chi_{(3,2)}} + \chi_{(0,4)}\overline{\chi_{(1,0)}} + \chi_{(0,1)}\overline{\chi_{(4,0)}} + \text{c.c}\big)\,.
\end{aligned} \tag{B.29}$$

Since this invariant is the $\mathbb{Z}_3$-orbifold of $su(3)_5$, one can obtain the boundary states for (B.29) by averaging the $\mathbb{Z}_3$-orbit of boundary states for the charge-conjugation invariant. For the untwisted states, this procedure yields the following coefficient matrix,

$$\Psi_{D^{(8)}} = \frac{1}{2\sqrt{2}} \begin{pmatrix}
1-\frac{1}{\sqrt{2}} & \sqrt{2} & 1+\frac{1}{\sqrt{2}} & 1 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \sqrt{2} & -\frac{1}{\sqrt{2}} & i & -i & i-\frac{1}{\sqrt{2}} & -i-\frac{1}{\sqrt{2}} \\
1 & 0 & 1 & -1+i & -1-i & -i & i \\
\sqrt{2} & 0 & -\sqrt{2} & 0 & 0 & \sqrt{2} & \sqrt{2} \\
1 & 0 & 1 & -1-i & -1+i & i & -i \\
\frac{1}{\sqrt{2}} & \sqrt{2} & -\frac{1}{\sqrt{2}} & -i & i & -i-\frac{1}{\sqrt{2}} & i-\frac{1}{\sqrt{2}} \\
1+\frac{1}{\sqrt{2}} & -\sqrt{2} & 1-\frac{1}{\sqrt{2}} & 1 & 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}, \tag{B.30}$$

where the columns are ordered as

$$\mathcal{E} = \{(0,0),(1,1),(2,2),(3,0),(0,3),(4,1),(1,4)\}. \tag{B.31}$$

Since all of three ω_c-twisted states for the charge-conjugation invariant are the fixed point of the $\mathbb{Z}_3$ action, we obtain $3 \times 3 = 9$ twisted states for (B.29) by an appropriate resolution of the fixed points. The resulting coefficient matrix takes the following form

$$\Psi_{D^{(8)*}} = \frac{1}{\sqrt{3}} \begin{pmatrix} \tau & \tau & \tau \\ \tau & e^{2\pi i/3}\,\tau & e^{-2\pi i/3}\,\tau \\ \tau & e^{-2\pi i/3}\,\tau & e^{2\pi i/3}\,\tau \end{pmatrix} \quad \text{with} \quad \tau = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} . \tag{B.32}$$

where the columns are ordered as

$$\mathcal{E}(\omega_c) = \{(0,0),(1,1),(2,2),(5,0),(3,1),(1,2),(0,5),(1,3),(2,1)\}. \tag{B.33}$$

In this way, we obtain

$$|\hat{\mathcal{V}}| = |\mathcal{V}| + |\mathcal{V}^{\omega_c}| = 7 + 9 = 16 \tag{B.34}$$

boundary states associated with the modular invariant (B.29). By calculating the overlap of these states, we obtain a set of 16×16 matrices

$$\begin{aligned} \hat{n}_{(0,0)} &= I_{16\times 16}, \\ \hat{n}_{(1,0)} &= \left(\begin{array}{c|c} A_{D^{(8)}} & \mathbf{O} \\ \hline \mathbf{O} & A_{D^{(8)*}} \end{array} \right) , \end{aligned} \tag{B.35}$$

where $A_{D^{(8)}}$ and $A_{D^{(8)*}}$ are the adjacency matrix for the graphs of type $D^{(8)}$ and $D^{(8)*}$, respectively,

$$A_{D^{(8)}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \; A_{D^{(8)*}} = \begin{pmatrix} O & O & \alpha \\ \alpha & O & O \\ O & \alpha & O \end{pmatrix}, \; \alpha = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} . \tag{B.36}$$

The remaining matrices for the untwisted representations take the following form,

$$
\hat{n}_{(2,0)} = \left(\begin{array}{c|ccc} A_{(2,0)} & & \mathbf{O} & \\ \hline & O & \alpha_{(2,0)} & O \\ \mathbf{O} & O & O & \alpha_{(2,0)} \\ & \alpha_{(2,0)} & O & O \end{array}\right), \ \hat{n}_{(1,1)} = \left(\begin{array}{c|ccc} A_{(1,1)} & & \mathbf{O} & \\ \hline & \alpha_{(1,1)} & O & O \\ \mathbf{O} & O & \alpha_{(1,1)} & O \\ & O & O & \alpha_{(1,1)} \end{array}\right),
$$

$$
\hat{n}_{(3,0)} = \left(\begin{array}{c|ccc} A^t_{(2,0)} & & \mathbf{O} & \\ \hline & \alpha_{(2,0)} & O & O \\ \mathbf{O} & O & \alpha_{(2,0)} & O \\ & O & O & \alpha_{(2,0)} \end{array}\right), \ \hat{n}_{(2,1)} = \left(\begin{array}{c|ccc} A_{(2,2)} & & \mathbf{O} & \\ \hline & O & O & \alpha_{(2,2)} \\ \mathbf{O} & \alpha_{(2,2)} & O & O \\ & O & \alpha_{(2,2)} & O \end{array}\right),
$$

$$
\hat{n}_{(4,0)} = \left(\begin{array}{c|ccc} A^t_{D(8)} & & \mathbf{O} & \\ \hline & O & O & \alpha \\ \mathbf{O} & \alpha & O & O \\ & O & \alpha & O \end{array}\right), \ \hat{n}_{(3,1)} = \left(\begin{array}{c|ccc} A_{(1,1)} & & \mathbf{O} & \\ \hline & O & \alpha_{(1,1)} & O \\ \mathbf{O} & O & O & \alpha_{(1,1)} \\ & \alpha_{(1,1)} & O & O \end{array}\right),
$$

$$
\hat{n}_{(2,2)} = \left(\begin{array}{c|ccc} A_{(2,2)} & & \mathbf{O} & \\ \hline & \alpha_{(2,2)} & O & O \\ \mathbf{O} & O & \alpha_{(2,2)} & O \\ & O & O & \alpha_{(2,2)} \end{array}\right), \ \hat{n}_{(5,0)} = \left(\begin{array}{c|ccc} I_{7\times 7} & & \mathbf{O} & \\ \hline & O & 1_{3\times 3} & O \\ \mathbf{O} & O & O & 1_{3\times 3} \\ & 1_{3\times 3} & O & O \end{array}\right),
$$

$$
\hat{n}_{(4,1)} = \left(\begin{array}{c|ccc} A_{D(8)} & & \mathbf{O} & \\ \hline & \alpha & O & O \\ \mathbf{O} & O & \alpha & O \\ & O & O & \alpha \end{array}\right), \ \hat{n}_{(3,2)} = \left(\begin{array}{c|ccc} A_{(2,0)} & & \mathbf{O} & \\ \hline & O & O & \alpha_{(2,0)} \\ \mathbf{O} & \alpha_{(2,0)} & O & O \\ & O & \alpha_{(2,0)} & O \end{array}\right), \tag{B.37}
$$

where

$$
A_{(2,0)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}, A_{(1,1)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}, A_{(2,2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}, \tag{B.38}
$$

$$
\alpha_{(2,0)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ \alpha_{(1,1)} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \ \alpha_{(2,2)} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}. \tag{B.39}
$$

For the twisted representations, we obtain

$$
\hat{n}_{(0)} = \left(\begin{array}{c|ccc} \mathbf{O} & B_0^t & B_0^t & B_0^t \\ \hline B_0 & & & \\ B_0 & & \mathbf{O} & \\ B_0 & & & \end{array}\right), \hat{n}_{(1)} = \left(\begin{array}{c|ccc} \mathbf{O} & B_1^t & B_1^t & B_1^t \\ \hline B_1 & & & \\ B_1 & & \mathbf{O} & \\ B_1 & & & \end{array}\right), \hat{n}_{(2)} = \left(\begin{array}{c|ccc} \mathbf{O} & B_2^t & B_2^t & B_2^t \\ \hline B_2 & & & \\ B_2 & & \mathbf{O} & \\ B_2 & & & \end{array}\right), \tag{B.40}
$$

where

$$B_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \quad \text{(B.41)}$$

We can check that these matrices form a NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_5; G_c)$ given in (B.8).

In summary, we have checked that the twisted boundary states of $su(3)_5$ realize a NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_5; G_c)$. In other words, we have constructed three NIM-reps of $\mathcal{F}(su(3)_5; G_c)$. Besides the regular one, which has dimension 24, we have obtained a 12-dimensional one associated to the exceptional modular invariant $Z_{E^{(8)}}$ and a 16-dimensional one associated to the simple current invariant $Z_{D^{(8)}}$.

C Twisted boundary states for $su(3)_1^{\oplus 3}$

In this appendix, we show that the S_3-twisted boundary states associated with the modular invariant (5.33) of $su(3)_1^{\oplus 3}$ yield a NIM-rep of the generalized fusion algebra $\mathcal{F}(su(3)_1^{\oplus 3}; S_3)$.

C.1 Generalized fusion algebra $\mathcal{F}(su(3)_1^{\oplus 3}; S_3)$

For $su(3)_1$, there are three integrable representations corresponding to three fundamental weights Λ_0, Λ_1 and Λ_2. In the following, we denote them as $(0) = (\Lambda_0), (1) = (\Lambda_1)$ and $(2) = (\Lambda_2)$. In this notation, the modular transformation matrix of $su(3)_1$ can be written as

$$S^{su(3)_1}_{mn} = \frac{1}{\sqrt{3}} e^{\frac{2\pi i}{3} mn} \quad (m, n = 0, 1, 2)\,. \tag{C.1}$$

Since this expression is invariant under the shift $m \to m+3$ (and also $n \to n+3$), we can consider that the representations of $su(3)_1$ are labeled by an integer modulo 3, namely, $(m+3) = (m)$. Actually, the fusion algebra of $su(3)_1$ is the group algebra of $\mathbb{Z}_3$,

$$(m) \times (n) = (m+n)\,. \tag{C.2}$$

In this notation, the action of the charge-conjugation is expressed as

$$(m)^* = (-m)\,. \tag{C.3}$$

The representations of $su(3)_1^{\oplus 3}$ are then labeled by three integers (n_1, n_2, n_3) and the set $\mathcal{I}$ of integrable representations reads

$$\mathcal{I} = \{(n_1, n_2, n_3) \,|\, n_i = 0, 1, 2\}\,. \tag{C.4}$$

Hence there are $3^3 = 27$ representations for $su(3)_1^{\oplus 3}$. The modular transformation matrix S is simply the tensor product of three $S^{su(3)_1}$,

$$S_{(m_1,m_2,m_3)(n_1,n_2,n_3)} = S^{su(3)_1}_{m_1n_1} S^{su(3)_1}_{m_2n_2} S^{su(3)_1}_{m_3n_3} = \frac{1}{3\sqrt{3}} e^{\frac{2\pi i}{3}(m_1n_1+m_2n_2+m_3n_3)}\,. \tag{C.5}$$

The fusion algebra of $su(3)_1^{\oplus 3}$ is the group algebra of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$,

$$(m_1, m_2, m_3) \times (n_1, n_2, n_3) = (m_1 + n_1, m_2 + n_2, m_3 + n_3)\,. \tag{C.6}$$

The automorphism group S_3 is generated by two elements π and σ, which act on the elements of $\mathcal{I}$ as

$$\begin{aligned} \pi &: (n_1, n_2, n_3) \mapsto (n_3, n_1, n_2)\,, \\ \sigma &: (n_1, n_2, n_3) \mapsto (n_1, n_3, n_2)\,. \end{aligned} \tag{C.7}$$

In terms of π and σ, the elements of S_3 can be expressed as follows,

$$S_3 = \{1, \pi, \pi^2, \sigma, \pi\sigma, \pi^2\sigma\}\,. \tag{C.8}$$

The fixed point of $\pi \in S_3$ is of the form (n, n, n),

$$\mathcal{I}(\pi) = \{(n, n, n)\,|\, n = 0, 1, 2\}\,. \tag{C.9}$$

Therefore we have three twining characters for π, which can be expressed by those of $su(3)_1$ as follows

$$\chi^{\pi}_{(n,n,n)}(\tilde{q}) = \mathrm{Tr}_{(n,n,n)}\, \pi\, \tilde{q}^{L_0 - \frac{c}{24}} = \chi^{su(3)_1}_n(\tilde{q}^3)\,. \tag{C.10}$$

As is shown in eq.(2.45), the π-twisted representations are obtained by the modular transformation of the twining characters χ^{π},

$$\chi^{su(3)_1}_m(q^{\frac{1}{3}}) = \sum_{n=0,1,2} S^{su(3)_1}_{mn} \chi^{su(3)_1}_n(\tilde{q}^3) = \sum_{n=0,1,2} S^{su(3)_1}_{mn} \chi^{\pi}_{(n,n,n)}(\tilde{q})\,. \tag{C.11}$$

Since the left-hand side is labeled by an integer $m\,(\mathrm{mod}\,3)$, we denote by $(m)_\pi$ the π-twisted representation of $su(3)_1^{\oplus 3}$,

$$\mathcal{I}^{\pi} = \{(m)_\pi\,|\, m = 0, 1, 2\}\,. \tag{C.12}$$

From the above equation, the characters and the modular transformation matrix for $(m)_\pi$ read

$$\chi_{(m)_\pi}(q) = \chi^{su(3)_1}_m(q^{\frac{1}{3}})\,, \quad S^{\pi}_{(m)_\pi (n,n,n)} = S^{su(3)_1}_{mn}\,. \tag{C.13}$$

For $\sigma \in S_3$, the fixed points and their twining characters have the form

$$\mathcal{I}(\sigma) = \{(n_1, n_2, n_2)\,|\, n_1, n_2 = 0, 1, 2\}\,, \tag{C.14}$$

$$\chi^{\sigma}_{(n_1,n_2,n_2)}(\tilde{q}) = \mathrm{Tr}_{(n_1,n_2,n_2)}\, \sigma\, \tilde{q}^{L_0 - \frac{c}{24}} = \chi^{su(3)_1}_{n_1}(\tilde{q})\chi^{su(3)_1}_{n_2}(\tilde{q}^2)\,. \tag{C.15}$$

We can therefore label the σ-twisted representations by a pair of integers,

$$\mathcal{I}^{\sigma} = \{(m_1, m_2)_\sigma\,|\, m_1, m_2 = 0, 1, 2\}\,, \tag{C.16}$$

for which the characters and the modular transformation matrix read

$$\chi_{(m_1,m_2)_\sigma}(q) = \chi^{su(3)_1}_{m_1}(q)\, \chi^{su(3)_1}_{m_2}(q^{\frac{1}{2}})\,, \quad S^{\sigma}_{(m_1,m_2)_\sigma (n_1,n_2,n_2)} = S^{su(3)_1}_{m_1 n_1}\, S^{su(3)_1}_{m_2 n_2}\,. \tag{C.17}$$

	(m_1,m_2,m_3)	$(m)_\pi$	$(m)_{\pi^2}$
(l_1,l_2,l_3)	$(l_1+m_1,l_2+m_2,l_3+m_3)$	$(l_1+l_2+l_3+m)_\pi$	$(l_1+l_2+l_3+m)_{\pi^2}$
$(l)_\pi$	$(l+m_1+m_2+m_3)_\pi$	$3(l+m)_{\pi^2}$	$\sum_{n_1+n_2+n_3=l+m}(n_1,n_2,n_3)$
$(l)_{\pi^2}$	$(l+m_1+m_2+m_3)_{\pi^2}$	$\sum_{n_1+n_2+n_3=l+m}(n_1,n_2,n_3)$	$3(l+m)_\pi$
$(l_1,l_2)_\sigma$	$(l_1+m_1,l_2+m_2+m_3)_\sigma$	$\sum_{n_1+n_2=l_1+l_2+m}(n_1,n_2)_{\pi^2\sigma}$	$\sum_{n_1+n_2=l_1+l_2+m}(n_1,n_2)_{\pi\sigma}$
$(l_1,l_2)_{\pi\sigma}$	$(l_1+m_1+m_2,l_2+m_3)_{\pi\sigma}$	$\sum_{n_1+n_2=l_1+l_2+m}(n_1,n_2)_\sigma$	$\sum_{n_1+n_2=l_1+l_2+m}(n_1,n_2)_{\pi^2\sigma}$
$(l_1,l_2)_{\pi^2\sigma}$	$(l_1+m_1+m_3,l_2+m_2)_{\pi^2\sigma}$	$\sum_{n_1+n_2=l_1+l_2+m}(n_1,n_2)_{\pi\sigma}$	$\sum_{n_1+n_2=l_1+l_2+m}(n_1,n_2)_\sigma$

	$(m_1,m_2)_\sigma$	$(m_1,m_2)_{\pi\sigma}$	$(m_1,m_2)_{\pi^2\sigma}$
(l_1,l_2,l_3)	$(l_1+m_1,l_2+l_3+m_2)_\sigma$	$(l_1+l_2+m_1,l_3+m_2)_{\pi\sigma}$	$(l_1+l_3+m_1,l_2+m_2)_{\pi^2\sigma}$
$(l)_\pi$	$\sum_{n_1+n_2=l+m_1+m_2}(n_1,n_2)_{\pi\sigma}$	$\sum_{n_1+n_2=l+m_1+m_2}(n_1,n_2)_{\pi^2\sigma}$	$\sum_{n_1+n_2=l+m_1+m_2}(n_1,n_2)_\sigma$
$(l)_{\pi^2}$	$\sum_{n_1+n_2=l+m_1+m_2}(n_1,n_2)_{\pi^2\sigma}$	$\sum_{n_1+n_2=l+m_1+m_2}(n_1,n_2)_\sigma$	$\sum_{n_1+n_2=l+m_1+m_2}(n_1,n_2)_{\pi\sigma}$
$(l_1,l_2)_\sigma$	$\sum_{n_2+n_3=l_2+m_2}(l_1+m_1,n_2,n_3)$	$(l_1+l_2+m_1+m_2)_{\pi^2}$	$(l_1+l_2+m_1+m_2)_\pi$
$(l_1,l_2)_{\pi\sigma}$	$(l_1+l_2+m_1+m_2)_\pi$	$\sum_{n_1+n_2=l_1+m_1}(n_1,n_2,l_2+m_2)$	$(l_1+l_2+m_1+m_2)_{\pi^2}$
$(l_1,l_2)_{\pi^2\sigma}$	$(l_1+l_2+m_1+m_2)_{\pi^2}$	$(l_1+l_2+m_1+m_2)_\pi$	$\sum_{n_1+n_3=l_1+m_1}(n_1,l_2+m_2,n_3)$

Table 4: Multiplication table of the generalized fusion algebra $\mathcal{F}(su(3)_1^{\oplus 3};S_3)$. The subscripts stand for the automorphism type of twisted representations. l_i,m_i and n_i take values in $\mathbb{Z}_3=\{0,1,2\}$. Since $\sigma\pi=\pi^2\sigma\neq\pi\sigma$, this algebra is non-commutative.

The remaining cases are treated in the same way and we give only the results below,

$$\mathcal{I}^{\pi^2}=\{(m)_{\pi^2}\,|\,m=0,1,2\}\,,\qquad S^{\pi^2}_{(m)_{\pi^2}(n,n,n)}=S^{su(3)_1}_{mn}\,,\tag{C.18}$$

$$\mathcal{I}^{\pi\sigma}=\{(m_1,m_2)_{\pi\sigma}\,|\,m_1,m_2=0,1,2\}\,,\qquad S^{\pi\sigma}_{(m_1,m_2)_{\pi\sigma}(n_1,n_1,n_2)}=S^{su(3)_1}_{m_1n_1}\,S^{su(3)_1}_{m_2n_2}\,,\tag{C.19}$$

$$\mathcal{I}^{\pi^2\sigma}=\{(m_1,m_2)_{\pi^2\sigma}\,|\,m_1,m_2=0,1,2\}\,,\qquad S^{\pi^2\sigma}_{(m_1,m_2)_{\pi^2\sigma}(n_1,n_2,n_1)}=S^{su(3)_1}_{m_1n_1}\,S^{su(3)_1}_{m_2n_2}\,.\tag{C.20}$$

The set $\hat{\mathcal{I}}$ of all the representations eventually consists of 60 elements,

$$|\hat{\mathcal{I}}|=|\mathcal{I}|+|\mathcal{I}^\pi|+|\mathcal{I}^{\pi^2}|+|\mathcal{I}^\sigma|+|\mathcal{I}^{\pi\sigma}|+|\mathcal{I}^{\pi^2\sigma}|=3^3+3\times 2+3^2\times 3=60\,.\tag{C.21}$$

From the formula (2.59), the conjugation acts on the twisted representations as follows,

$$\begin{aligned}(m_1,m_2,m_3)^*&=(-m_1,-m_2,-m_3)\,,\\ (m)^*_\pi&=(-m)_{\pi^2}\,,\\ (m_1,m_2)^*_\omega&=(-m_1,-m_2)_\omega\quad(\omega\in\{\sigma,\pi\sigma,\pi^2\sigma\})\,.\end{aligned}\tag{C.22}$$

Having obtained twisted representations and their modular transformation matrices, it is straightforward to calculate the generalized fusion coefficients of $\mathcal{F}(su(3)_1^{\oplus 3};S_3)$ using the

formula (2.62). For example, the coefficient $\widehat{\mathcal{N}}_{(l_1,l_2)_\sigma (m)_\pi}{}^{(n_1,n_2)_{\pi^2\sigma}}$ can be obtained as follows,

$$\begin{aligned}\widehat{\mathcal{N}}_{(l_1,l_2)_\sigma (m)_\pi}{}^{(n_1,n_2)_{\pi^2\sigma}} &= \sum_{\lambda\in\mathcal{I}(\sigma)\cap\mathcal{I}(\pi)} \frac{S^{\sigma}_{(l_1,l_2)_\sigma\,\lambda}\, S^{\pi}_{(m)_\pi\,\lambda}\, \overline{S^{\pi^2\sigma}_{(n_1,n_2)_{\pi^2\sigma}\,\lambda}}}{S_{(0,0,0)\lambda}} \\ &= \sum_{p=0,1,2} \frac{S^{\sigma}_{(l_1,l_2)_\sigma (p,p,p)}\, S^{\pi}_{(m)_\pi (p,p,p)}\, \overline{S^{\pi^2\sigma}_{(n_1,n_2)_{\pi^2\sigma}(p,p,p)}}}{S_{(0,0,0)(p,p,p)}} \\ &= \sum_{p=0,1,2} \frac{S^{su(3)_1}_{l_1p} S^{su(3)_1}_{l_2p} S^{su(3)_1}_{mp} \overline{S^{su(3)_1}_{n_1p} S^{su(3)_1}_{n_2p}}}{(S^{su(3)_1}_{0p})^3} \\ &= \frac{1}{3}\sum_{p=0,1,2} e^{\frac{2\pi i}{3}p(l_1+l_2+m-n_1-n_2)} \\ &= \delta^{(3)}_{l_1+l_2+m,\, n_1+n_2}\,,\end{aligned} \tag{C.23}$$

where $\delta^{(3)}$ is the Kronecker delta for $\mathbb{Z}_3$. The other cases can be calculated in the same manner; we give the result in Table 4.

C.2 Non-trivial NIM-rep of $\mathcal{F}(su(3)_1^{\oplus 3}; S_3)$

For the chiral algebra $su(3)_1^{\oplus 3}$, the following block diagonal invariant is available,

$$\begin{aligned} Z_{E_6} = &\left|\chi_{(0,0,0)} + \chi_{(1,1,1)} + \chi_{(2,2,2)}\right|^2 \\ &+ \left|\chi_{(0,2,1)} + \chi_{(1,0,2)} + \chi_{(2,1,0)}\right|^2 + \left|\chi_{(0,1,2)} + \chi_{(2,0,1)} + \chi_{(1,2,0)}\right|^2 ,\end{aligned} \tag{C.24}$$

which originates from the conformal embedding $su(3)_1^{\oplus 3} \subset E_{6,1}$ (see Fig. 3). [13] For $E_{6,1}$, there are three integrable representations, $(\tilde{\Lambda}_0)$, $(\tilde{\Lambda}_1)$ and $(\tilde{\Lambda}_5)$, [14] and the charge-conjugation invariant reads

$$Z = |\chi_{\tilde{\Lambda}_0}|^2 + |\chi_{\tilde{\Lambda}_1}|^2 + |\chi_{\tilde{\Lambda}_5}|^2 . \tag{C.25}$$

From this together with the branching rule

$$\begin{aligned}(\tilde{\Lambda}_0) &= (0,0,0)\oplus(1,1,1)\oplus(2,2,2)\,, \\ (\tilde{\Lambda}_1) &= (0,2,1)\oplus(1,0,2)\oplus(2,1,0)\,, \\ (\tilde{\Lambda}_5) &= (0,1,2)\oplus(1,2,0)\oplus(2,0,1)\,,\end{aligned} \tag{C.26}$$

one obtains the invariant (C.24).

There are three boundary states preserving $E_{6,1}$ corresponding to three integrable representations, [15]

$$|\tilde{\Lambda}_i\rangle = \sum_{j=0,1,5} S^{E_6}_{\tilde{\Lambda}_i\tilde{\Lambda}_j} |\tilde{\Lambda}_j\rangle\!\rangle \quad (i=0,1,5)\,. \tag{C.27}$$

[13] This invariant is also considered to be a simple current extension by $(1,1,1)\in\mathcal{I}$.

[14] We distinguish the fundamental weights of $E_{6,1}$ from those of $su(3)_1$ by putting a tilde.

[15] For simplicity, we omit in this appendix the label for the anti-holomorphic representation of Ishibashi states and denote by $|\lambda;\omega\rangle\!\rangle$ instead of $|(\lambda,\mu^*);\omega\rangle\!\rangle$.

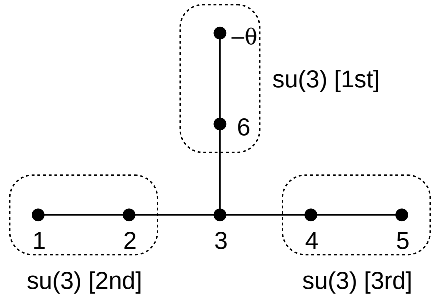

Figure 3: Extended Dynkin diagram of E_6. θ stands for the highest root of E_6. Each box expresses the simple roots of $su(3) \subset E_6$.

S^{E_6} is the modular transformation matrix of $E_{6,1}$

$$S^{E_6} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \tag{C.28}$$

where the rows and the columns are ordered as $\{(\tilde{\Lambda}_0), (\tilde{\Lambda}_1), (\tilde{\Lambda}_5)\}$. From the branching rule (C.26), one can express the Ishibashi states of $E_{6,1}$ in terms of those of $su(3)_1^{\oplus 3}$,

$$\begin{aligned} |\tilde{\Lambda}_0\rangle\rangle &= \frac{1}{\sqrt{3}}(|(0,0,0)\rangle\rangle + |(1,1,1)\rangle\rangle + |(2,2,2)\rangle\rangle), \\ |\tilde{\Lambda}_1\rangle\rangle &= \frac{1}{\sqrt{3}}(|(0,2,1)\rangle\rangle + |(1,0,2)\rangle\rangle + |(2,1,0)\rangle\rangle), \\ |\tilde{\Lambda}_5\rangle\rangle &= \frac{1}{\sqrt{3}}(|(0,1,2)\rangle\rangle + |(1,2,0)\rangle\rangle + |(2,0,1)\rangle\rangle). \end{aligned} \tag{C.29}$$

Substituting this into (C.27), one obtains three boundary states preserving $su(3)_1^{\oplus 3}$. The remaining 6 $(= 9 - 3)$ states can be constructed by the fusion with $(1,0,0) \in \mathcal{I}$. Since $(1,0,0)^3 = (3,0,0) = (0,0,0)$, we obtain three boundary states for each state preserving $E_{6,1}$. We therefore label the resulting states as follows

$$\begin{aligned} \mathcal{V} = \{&(0,0) = (\tilde{\Lambda}_0), (0,1), (0,2), \\ &(1,0) = (\tilde{\Lambda}_1), (1,1), (1,2), (2,0) = (\tilde{\Lambda}_5), (2,1), (2,2)\}. \end{aligned} \tag{C.30}$$

The boundary state coefficient takes the form

$$\Psi = \frac{1}{\sqrt{3}} \begin{pmatrix} K & K & K \\ K & e^{-\frac{2\pi i}{3}} K & e^{\frac{2\pi i}{3}} K \\ K & e^{\frac{2\pi i}{3}} K & e^{-\frac{2\pi i}{3}} K \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \end{pmatrix} \otimes K, \tag{C.31}$$

where K is a 3×3 unitary matrix

$$K = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \\ 1 & e^{-\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \end{pmatrix}. \tag{C.32}$$

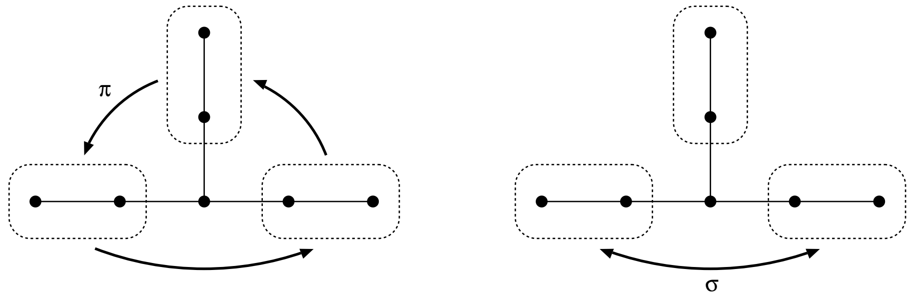

Figure 4: The automorphism group S_3 of $su(3)^{\oplus 3}$ has a lift to E_6.

The row of Ψ is ordered as (C.30) while the column is ordered as

$$\mathcal{E} = \{(0,0,0),(1,1,1),(2,2,2),(0,2,1),(1,0,2),(2,1,0),(0,1,2),(1,2,0),(2,0,1)\}\,. \quad \text{(C.33)}$$

The overlap matrices $\hat{n}$ can be calculated using the formula (4.13), and the result is

$$(\hat{n}_{(m_1,m_2,m_3)})_{(\alpha,a)}{}^{(\beta,b)} = (P^{m_2+2m_3})_\alpha{}^\beta \otimes (P^{m_1+m_2+m_3})_a{}^b\,, \quad \text{(C.34)}$$

where $(\alpha,a),(\beta,b) \in \mathcal{V}$ and P is a 3×3 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\,. \quad \text{(C.35)}$$

These matrices $\{\hat{n}_{(m_1,m_2,m_3)} | (m_1,m_2,m_3) \in \mathcal{I}\}$ satisfy the ordinary fusion algebra $\mathcal{F}(su(3)_1^{\oplus 3})$

$$\hat{n}_{(l_1,l_2,l_3)}\,\hat{n}_{(m_1,m_2,m_3)} = P^{l_2+2l_3+m_2+2m_3} \otimes P^{l_1+l_2+l_3+m_1+m_2+m_3} = \hat{n}_{(l_1+m_1,l_2+m_2,l_3+m_3)}\,, \quad \text{(C.36)}$$

which means that the untwisted boundary states (C.31) form a NIM-rep of $\mathcal{F}(su(3)_1^{\oplus 3})$.

We turn to the construction of twisted boundary states in the invariant (C.24). The automorphism group S_3 of $su(3)_1^{\oplus 3}$ has a lift to $E_{6,1}$ (see Fig. 4). The automorphism group $\mathrm{Aut}(E_6)$ of E_6 has a normal subgroup $\mathrm{Aut}_0(E_6)$ consisting of all the inner automorphisms. The quotient group $\mathrm{Aut}(E_6)/\mathrm{Aut}_0(E_6)$ has two elements: one is the identity and corresponds to $\mathrm{Aut}_0(E_6)$ while the other comes from the outer automorphisms that contains the charge conjugation $\tilde{\omega}_c$ of E_6,

$$\tilde{\omega}_c : (\tilde{\Lambda}_0) \mapsto (\tilde{\Lambda}_0)\,, \quad (\tilde{\Lambda}_1) \leftrightarrow (\tilde{\Lambda}_5)\,. \quad \text{(C.37)}$$

As is seen from the definition (C.7) of $\pi \in S_3$ and the branching rule (C.26), π does not change any representation of E_6. Hence the lift of π to E_6 is an inner automorphism of E_6. On the other hand, $\sigma \in S_3$ exchanges $(\tilde{\Lambda}_1)$ with $(\tilde{\Lambda}_5)$ and its lift is an outer automorphism of E_6.

Let $\tilde{\pi}$ be the lift of π to E_6. Since $\tilde{\pi}$ is inner, the corresponding twisted boundary states of E_6 are expressed by the same boundary state coefficient as the untwisted ones, namely the modular transformation matrix (C.28). Therefore, applying the fusion $(1,0,0) \in \mathcal{I}$ to

the $\tilde{\pi}$-twisted states, we obtain exactly the same boundary state coefficient for the π-twisted states as the untwisted ones,

$$\Psi^{\pi} = \Psi\,. \tag{C.38}$$

Accordingly the labels of the π-twisted boundary states has the same structure as $\mathcal{V}$,

$$\mathcal{V}^{\pi} = \{(\alpha, a)_{\pi} \,|\, \alpha = 0, 1, 2;\; a = 0, 1, 2\}\,, \quad \mathcal{E}(\pi) = \mathcal{E}\,. \tag{C.39}$$

We can construct the π^2-twisted states in the same way as the case of π and obtain the result

$$\Psi^{\pi^2} = \Psi\,, \quad \mathcal{V}^{\pi^2} = \{(\alpha, a)_{\pi^2} \,|\, \alpha = 0, 1, 2;\; a = 0, 1, 2\}\,, \quad \mathcal{E}(\pi^2) = \mathcal{E}\,. \tag{C.40}$$

The lift $\tilde{\sigma}$ of $\sigma \in S_3$ to E_6 is outer. Hence we have to start from a non-trivial boundary state coefficient in E_6 instead of the modular transformation matrix. Since $\tilde{\sigma}$ fixes only $(\tilde{\Lambda}_0)$ among the integrable representations of $E_{6,1}$, there is only one $\tilde{\sigma}$-twisted boundary state [16]

$$|(0)_{\sigma}\rangle = |\tilde{\Lambda}_0; \tilde{\sigma}\rangle\!\rangle = \frac{1}{\sqrt{3}}(|(0,0,0);\sigma\rangle\!\rangle + |(1,1,1);\sigma\rangle\!\rangle + |(2,2,2);\sigma\rangle\!\rangle)\,. \tag{C.41}$$

Applying the fusion with $(1, 0, 0) \in \mathcal{I}$ yields the remaining two states. The result is as follows,

$$\Psi^{\sigma} = K\,, \quad \mathcal{V}^{\sigma} = \{(a)_{\sigma} \,|\, a = 0, 1, 2\}\,, \quad \mathcal{E}(\sigma) = \{(0,0,0), (1,1,1), (2,2,2)\}\,, \tag{C.42}$$

where K is the matrix defined in (C.32). The case of $\pi\sigma$ and $\pi^2\sigma$ can be treated in the same way as σ and yields the result

$$\Psi^{\pi\sigma} = K\,, \qquad \mathcal{V}^{\pi\sigma} = \{(a)_{\pi\sigma} \,|\, a = 0, 1, 2\}\,, \qquad \mathcal{E}(\pi\sigma) = \mathcal{E}(\sigma)\,, \tag{C.43}$$

$$\Psi^{\pi^2\sigma} = K\,, \qquad \mathcal{V}^{\pi^2\sigma} = \{(a)_{\pi^2\sigma} \,|\, a = 0, 1, 2\}\,, \qquad \mathcal{E}(\pi^2\sigma) = \mathcal{E}(\sigma)\,. \tag{C.44}$$

In this way, we obtain 36 boundary states in the block diagonal invariant (C.24),

$$|\hat{\mathcal{V}}| = |\mathcal{V}| + |\mathcal{V}^{\pi}| + |\mathcal{V}^{\pi^2}| + |\mathcal{V}^{\sigma}| + |\mathcal{V}^{\pi\sigma}| + |\mathcal{V}^{\pi^2\sigma}| = 9 \times 3 + 3 \times 3 = 36\,. \tag{C.45}$$

From the boundary state coefficients obtained above, we can calculate the overlap matrices $\hat{n}$ by the formula (4.13). In expressing $\hat{n}$, which is a 36×36 matrix, it is convenient to factorize $\hat{\mathcal{V}}$ in the manner similar to (C.34),

$$\hat{\mathcal{V}} = \{(0), (1), (2), (0)_{\pi}, (1)_{\pi}, (2)_{\pi}, (0)_{\pi^2}, (1)_{\pi^2}, (2)_{\pi^2}, (0)_{\sigma}, (0)_{\pi\sigma}, (0)_{\pi^2\sigma}\} \otimes \{(a) \,|\, a = 0, 1, 2\}\,, \tag{C.46}$$

[16] One can regard the boundary state coefficient of $|(0)_{\sigma}\rangle$ as the modular transformation 'matrix' of the twisted chiral algebra $E_6^{(2)}$ at level 1.

which is related to the original notation as, *e.g.*, $(\alpha)_\pi \otimes (a) = (\alpha, a)_\pi \in \mathcal{V}^\pi$, $(0)_\sigma \otimes (a) = (a)_\sigma \in \mathcal{V}^\sigma$. In this basis, one can show that the matrices $\hat{n}$ take the following form,

$$\begin{aligned}
\hat{n}_{(m_1,m_2,m_3)} &= \begin{pmatrix} P^{m_2+2m_3} & O & O & O \\ O & P^{m_2+2m_3} & O & O \\ O & O & P^{m_2+2m_3} & O \\ O & O & O & I \end{pmatrix} \otimes P^{m_1+m_2+m_3}\,, \\
\hat{n}_{(m)_\pi} &= \begin{pmatrix} O & T & O & O \\ O & O & T & O \\ T & O & O & O \\ O & O & O & 3P^2 \end{pmatrix} \otimes P^m\,, \quad \hat{n}_{(m)_{\pi^2}} = \begin{pmatrix} O & O & T & O \\ T & O & O & O \\ O & T & O & O \\ O & O & O & 3P \end{pmatrix} \otimes P^m\,, \\
\hat{n}_{(m_1,m_2)_\sigma} &= \begin{pmatrix} O & O & O & E_1 \\ O & O & O & E_2 \\ O & O & O & E_3 \\ E_1^T & E_2^T & E_3^T & O \end{pmatrix} \otimes P^{m_1+m_2}\,,\ \hat{n}_{(m_1,m_2)_{\pi\sigma}} = \begin{pmatrix} O & O & O & E_2 \\ O & O & O & E_3 \\ O & O & O & E_1 \\ E_2^T & E_3^T & E_1^T & O \end{pmatrix} \otimes P^{m_1+m_2}\,, \\
\hat{n}_{(m_1,m_2)_{\pi^2\sigma}} &= \begin{pmatrix} O & O & O & E_3 \\ O & O & O & E_1 \\ O & O & O & E_2 \\ E_3^T & E_1^T & E_2^T & O \end{pmatrix} \otimes P^{m_1+m_2}\,,
\end{aligned} \tag{C.47}$$

where P is a permutation matrix of (C.35) while O and I are the zero and the unit matrices, respectively. The matrices T, E_1, E_2 and E_3 are defined as follows,

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\,, \quad E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{C.48}$$

We have checked that these 60 matrices $\{\hat{n}_N | N \in \hat{\mathcal{I}}\}$ satisfy the generalized fusion algebra $\mathcal{F}(su(3)_1^{\oplus 3}; S_3)$. For example, $(0,0)_\sigma \times (0)_\pi = (0,0)_{\pi^2\sigma} + (1,2)_{\pi^2\sigma} + (2,1)_{\pi^2\sigma}$ is satisfied as follows

$$\begin{aligned}
\hat{n}_{(0,0)_\sigma}\, \hat{n}_{(0)_\pi} &= \begin{pmatrix} O & O & O & E_1 \\ O & O & O & E_2 \\ O & O & O & E_3 \\ E_1^T & E_2^T & E_3^T & O \end{pmatrix} \begin{pmatrix} O & T & O & O \\ O & O & T & O \\ T & O & O & O \\ O & O & O & 3P^2 \end{pmatrix} \otimes I \\
&= \begin{pmatrix} O & O & O & 3E_3 \\ O & O & O & 3E_1 \\ O & O & O & 3E_2 \\ 3E_3^T & 3E_1^T & 3E_2^T & O \end{pmatrix} \otimes I = \hat{n}_{(0,0)_{\pi^2\sigma}} + \hat{n}_{(1,2)_{\pi^2\sigma}} + \hat{n}_{(2,1)_{\pi^2\sigma}}\,.
\end{aligned} \tag{C.49}$$

The matrices $\{\hat{n}_N\}$ therefore form a 36-dimensional NIM-rep of $\mathcal{F}(su(3)_1^{\oplus 3}; S_3)$. Together with the regular NIM-rep, which is 60-dimensional, we have obtained two NIM-reps of $\mathcal{F}(su(3)_1^{\oplus 3}; S_3)$ corresponding to two modular invariants of $su(3)_1^{\oplus 3}$.

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